A NONSMOOTH PRIMAL-DUAL METHOD WITH SIMULTANEOUS ADAPTIVE PDE CONSTRAINT SOLVER

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Abstract We introduce an efficient first-order primal-dual method for the solution of nonsmooth PDE-constrained optimization problems. We achieve this efficiency through *not* solving the PDE or its linearisation on each iteration of the optimization method. Instead, we run the method in parallel with a simple conventional linear system solver (Jacobi, Gauss–Seidel, conjugate gradients), always taking only *one step* of the linear system solver for each step of the optimization method. The control parameter is updated on each iteration as determined by the optimization method. We prove linear convergence under a second-order growth condition, and numerically demonstrate the performance on a variety of PDEs related to inverse problems involving boundary measurements.

1 INTRODUCTION

Our objective is to develop efficient first-order algorithms for the solution of PDE-constrained optimization problems of the type

$$\min_{x,u} F(x) + Q(u) + G(Kx) \quad \text{subject to} \quad B(u, w; x) = Lw \quad \text{for all} \quad w,$$

where K is a linear operator and the functions F, G, and Q are convex but the first two possibly nonsmooth. The functionals B and L model a partial differential equation in weak form, parametrised by x; for example, $B(u, w; x) = \langle \nabla u, x\nabla w\rangle$.

Semismooth Newton methods [22, 23] are conventionally used for such problems when a suitable reformulation exists [16, 18, 27, 28, 17]. Reformulations may not always be available, or yield effective algorithms. The solution of large linear systems may also pose scaling challenges. Therefore, first-order methods for PDE-constrained optimization have been proposed [6, 4, 21, 5] based on the primal-dual proximal splitting (PDPS) of [2]. The original version applies to convex problems of the form

$$\min_{x} F(x) + G(Kx). \tag{1.1}$$

The primal-dual expansion permits efficient treatment of $G \circ K$ for nonsmooth G. In [6, 4, 21, 5] K may be nonlinear, such as the solution operator of a nonlinear PDE.

However, first-order methods generally require a very large number of iterations to exhibit convergence. If the iterations are cheap, they can, nevertheless, achieve good performance. If the iterations are expensive, such as when a PDE needs to be solved on each step, their performance can be poor. Therefore, especially in inverse problems research, Gauss–Newton -type approaches are common for (1.1) with nonlinear K; see, e.g., [8, 31, 19]. They are easy: first linearise K, then apply a convex optimization

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method or, in simplest cases, a linear system solver. Repeat. Even when a first-order method is used for the subproblem, Gauss–Newton methods can be significantly faster than full first-order methods [19] if they converge at all [29]. This stems from the following and only practical difference between the PDPS for nonlinear K and Gauss–Newton applied to (1.1) with PDPS for the inner problems: the former re-linearizes and factors K on *each* PDPS iteration, the latter only on each outer Gauss–Newton iteration.

In this work, we avoid forming and factorizing the PDE solution operators altogether by *running an iterative solver for the constantly adapting PDE simultaneously with the optimization method.* This may be compared to the approach to bilevel optimization in [25]. We concentrate on the simple Jacobi and Gauss–Seidel splitting methods for the PDE, while the optimization method is based on the PDPS, as we describe in Section 2. We prove convergence in Section 3 using the testing approach introduced in [30] and further elucidated in [7]. We explain how standard splittings and PDEs fit into the framework in Section 4, and finish with numerical experiments in Section 5.

Pseudo-time-stepping one-shot methods have been introduced in [26] and further studied, among others, in [24, 20, 13, 12, 11, 1, 10, 14]. A "one-shot" approach, as opposed to an "all-at-once" approach, solves the PDE constraints on each step, instead of considering them part of a unified system of optimality conditions. The aforementioned works solve these constraints inexactly through "pseudo-"time-stepping. This corresponds to the trivial split $A_x = (A_x - \mathrm{Id}) + \mathrm{Id}$ where A_x is such that $\langle A_x u, w\rangle = B(u, w; x)$. We will, instead, apply Jacobi, Gauss–Seidel or even (quasi-)conjugate gradient splitting on A_x. In [10, 1] Jacobi and Gauss–Seidel updates are used for the control variable, but not for the PDEs. The authors of [14] come closest to introducing non-trivial splitting of the PDEs via Hessian approximation. However, they and the other aforementioned works generally restrict themselves to smooth problems and employ gradient descent, Newton-type methods, or sequential quadratic programming (SQP) for the control variable x. Our focus is on nonsmooth problems involving, in particular, total variation regularization $G(Kx) = \|\nabla x\|_1$.

NOTATION AND BASIC RESULTS

Let X be a normed space. We write $\langle\,\cdot\,|\,\cdot\,\rangle$ for the dual product and, in a Hilbert space, $\langle\,\cdot\,,\,\cdot\,\rangle$ for the inner product. When we identify Hilbert X with X^*, we also assume $\langle\,\cdot\,|\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle$. We write $\mathbb{L}(X;Y)$ for the space of bounded linear operators between X and Y, and abbreviate $\mathbb{L}(X) := \mathbb{L}(Y;\mathbb{R})$. We write $\mathrm{Id}_X = \mathrm{Id} \in \mathbb{L}(X;X)$ for the identity operator on X. If $M \in \mathbb{L}(X;X^*)$ is non-negative and self-adjoint, i.e., $\langle Mx|y\rangle = \langle x|My\rangle$ and $\langle x|Mx\rangle \geq 0$ for all $x, y \in X$, we define $\|x\|_M := \sqrt{\langle x|Mx\rangle}$. Then the *three-point identity* holds:

$$\langle x - y|x - z\rangle_M = \frac{1}{2}\|x - y\|_M^2 - \frac{1}{2}\|y - z\|_M^2 + \frac{1}{2}\|x - z\|_M^2 \qquad \text{for all } x, y, z \in X. \tag{1.2}$$

We extensively use the vector Young's inequality

$$\langle x|y\rangle \leq \frac{1}{2a}\|x\|_X^2 + \frac{a}{2}\|y\|_Y^2 \quad (x \in X,\ y \in Y,\ a > 0). \tag{1.3}$$

These expressions hold in Hilbert spaces also with the inner product in place of the dual product. We write M^* for the inner product adjoint of M.

We write $\operatorname{dom} F$ for the effective domain, and F^* for the Fenchel conjugate of $F : X \to \overline{\mathbb{R}} := [-\infty, \infty]$. We write $F'(x) \in X^*$ for the Fréchet derivative at x when it exists, and, if X is Hilbert, $\nabla F(x) \in X$ for its Riesz presentation. For convex F, we write $\partial F(x) \subset X^*$ for the subdifferential at $x \in X$. When X is Hilbert we identify $\partial F(x)$ with the set of Riesz representations of its elements in X. We define the proximal map

$$\mathrm{prox}_F(x) := (\mathrm{Id} + \partial F)^{-1}(x) = \arg\min_{\tilde{x}} \left\{ F(\tilde{x}) + \frac{1}{2}\|\tilde{x} - x\|^2 \right\}.$$

We occasionally apply operations on $x \in X$ to all elements of sets $A \subset X$, writing $\langle x + A|z\rangle := \{\langle x + a|z\rangle \mid a \in A\}$. For $B \subset \mathbb{R}$, we write $B \geq c$ if $b \geq c$ for all $b \in B$.

On a Lipschitz domain $\Omega \subset \mathbb{R}^n$, we write $\operatorname{trace}_{\partial\Omega} \in \mathbb{L}(H^1(\Omega); L^2(\partial\Omega))$ for the trace operator on the boundary $\partial\Omega$.

2 PROBLEM AND PROPOSED ALGORITHM

We start by introducing in detail the type of problem we are trying to solve. We then rewrite in Section 2.1 its optimality conditions in a form suitable for developing our proposed method in Section 2.3. Before this we recall the structure and derivation of the basic PDPS in Section 2.2.

2.1 PROBLEM DESCRIPTION

Our objective is to solve

$$\min_x J(x) := F(x) + Q(S(x)) + G(Kx), \tag{2.1}$$

where $F : X \to \overline{\mathbb{R}}$, $G : Y \to \overline{\mathbb{R}}$, and $Q : U \to \mathbb{R}$ are convex, proper, and lower semicontinuous on Hilbert spaces X, U, and Y with Q Fréchet differentiable. We assume $K \in \mathbb{L}(X; Y)$ while $S : X \ni x \mapsto u \in U$ is a solution operator of the weak PDE

$$B(u, w; x) = Lw \quad \text{for all} \quad w \in W. \tag{2.2}$$

Here $L \in \mathbb{L}(U)$ and $B : U \times W \times X \to \mathbb{R}$ is bilinear in u and w, and affine in x. The space W is Hilbert, possibly distinct from U to model homogeneous boundary conditions.

Example 2.1. On a Lipschitz domain $\Omega \subset \mathbb{R}^n$, consider the PDE

$$\begin{cases} \nabla \cdot (x\nabla u) = 0, & \text{on } \Omega, \\ u = g, & \text{on } \partial\Omega \end{cases}$$

For the weak form (2.2) we can take the spaces $U = H^1(\Omega)$, $W = H^1_0(\Omega) \times H^{1/2}(\Omega)$, and $X = L^2(\Omega)$. Writing $w = (w_\Omega, w_\partial)$, we then set

$$B(u, w; x) = \langle x\nabla u, \nabla w_\Omega\rangle_{L^2(\Omega)} + \langle \operatorname{trace}_{\partial\Omega} u, w_\partial\rangle_{L^2(\partial\Omega)} \quad \text{and} \quad Lw := \langle g, w_\partial\rangle_{L^2(\partial\Omega)}.$$

We require the sum and chain rules for convex subdifferentials to hold on $F + G \circ K$. This is the case when

$$\text{there exists an } x \in \operatorname{dom}(G \circ K) \cap \operatorname{dom} F \text{ with } Kx \in \operatorname{int}(\operatorname{dom} G). \tag{2.3}$$

We refer to [7] for basic results and concepts of infinite-dimensional convex analysis. Then by the Fréchet differentiability of Q and the compatibility of Clarke subdifferentials (denoted ∂_C) with Fréchet derivatives and convex subdifferentials [3, 7],

$$\partial_C J(x) = \partial F(x) + \nabla S(x)^* \nabla Q(S(x)) + K^* \partial G(Kx)$$

Therefore, the Fermat principle for Clarke subdifferentials and simple rearrangements (see [29, 4] or [7, Chapter 15]) establish for (2.1) in terms of $(\bar{u}, \bar{w}, \bar{x}, \bar{y}) \in U \times W \times X \times Y$ the necessary first-order optimality condition

$$\begin{cases} \bar{u} = S(\bar{x}), \\ -\nabla S(\bar{x})^* \nabla Q(\bar{u}) - K^* \bar{y} \in \partial F(\bar{x}), \\ K\bar{x} \in \partial G^*(\bar{y}). \end{cases} \tag{2.4}$$

We recall that G^* is the Fenchel conjugate of G.

The term $\nabla S(\bar{x})^*\nabla Q(\bar{u})$ involves the solution $\bar{u}$ to the original PDE and the solution $\bar{w}$ to an adjoint PDE. We derive it from a primal-dual reformulation of (2.1). To do this, we first observe that since B is affine in x, it can be decomposed as

$$B(u, w; x) = B_{\text{lin}}(u, w; x) + B_{\text{const}}(u, w), \tag{2.5}$$

where $B_{\text{lin}} : U \times W \times X \to \mathbb{R}$ is trilinear, and $B_{\text{const}} : U \times W \to \mathbb{R}$ is bilinear. Indeed $B_{\text{const}}(u, w) = B(u, w; 0)$, and $B_{\text{lin}}(u, w; x) = B(u, w; x) - B(u, w; 0)$. We then introduce the Riesz representation $B_x(u, w)$ of $B_{\text{lin}}(u, w; \,\cdot\,) \in X^*$. Thus

$$\langle B_x(u, w), x\rangle_X = B_{\text{lin}}(u, w; x) \quad \text{for all } u \in U,\ w \in W,\ x \in X. \tag{2.6}$$

Then $\nabla_x B(u, w; x) \equiv B_x(u, w) \in X$ for all $x \in X$ while the (non-Riesz-represented) Fréchet derivatives with respect to u and w are $B(\,\cdot\,, w; x) \in U^*$ and $B(u, \,\cdot\,; x) \in W^*$.

We may now write (2.1) as

$$\min_{x,u} \max_{w}\ F(x) + Q(u) + B(u, w; x) - Lw + G(Kx)$$

or

$$\min_{x,u} \max_{w,y}\ F(x) + Q(u) + B(u, w; x) - Lw + \langle Kx, y\rangle_Y - G^*(y). \tag{2.7}$$

Similarly to the derivation of (2.4), in terms of $(\bar{u}, \bar{w}, \bar{x}, \bar{y}) \in U \times W \times X \times Y$, this problem has the necessary first-order optimality conditions

$$\begin{cases} B(\bar{u}, \tilde{w}; \bar{x}) = L\tilde{w} & \text{for all} \quad \tilde{w} \in W, \\ B(\tilde{u}, \bar{w}; \bar{x}) = -Q'(\bar{u})\tilde{u} & \text{for all} \quad \tilde{u} \in U, \\ -B_x(\bar{u}, \bar{w}) - K^*\bar{y} \in \partial F(\bar{x}), \\ K\bar{x} \in \partial G^*(\bar{y}). \end{cases} \tag{2.8}$$

This is our principal form of optimal conditions for (2.1).

The second line of (2.8) is the adjoint PDE, needed for $\nabla S(\bar{x})^*\nabla Q(\bar{u})$ in (2.4):

Lemma 2.2. *The solution operator S of (2.2) satisfies for all $x, z \in X$ that*

$$\nabla S(x)^* z = B_x(u, w) \quad \textit{where} \quad u = S(x) \quad \textit{and} \quad \begin{cases} w \textit{ solves the weak adjoint PDE:} \\ B(\tilde{u}, w; x) = -\langle z, \tilde{u}\rangle \textit{ for all } \tilde{u} \in U. \end{cases}$$

Proof. Comparing (2.4) to (2.8), we see that $\nabla S(\bar{x})^*\nabla Q(\bar{u}) = B_x(\bar{u}, \bar{w})$ for $\bar{w} \in W$ solving $B(\tilde{u}, \bar{w}; \bar{x}) = -Q'(\bar{u})\tilde{u}$ for all $\tilde{u} \in U$. Taking $Q(u) = \langle z, u\rangle$, the claim follows. □

2.2 PRIMAL-DUAL PROXIMAL SPLITTING: A RECAP

The primal-dual proximal splitting (PDPS) for (1.1) is based on the optimality conditions

$$\begin{cases} -K^*\bar{y} \in \partial F(\bar{x}), \\ \quad K\bar{x} \in \partial G^*(\bar{y}). \end{cases} \tag{2.9}$$

These are just the last two lines of (2.8) without B_x. As derived in [30, 15, 7], the basic (unaccelerated) PDPS solves (2.9) by iteratively solving for each $k \in \mathbb{N}$ the system

$$\begin{cases} 0 \in \tau\partial F(x^{k+1}) + \tau K^* y^k + x^{k+1} - x^k \\ 0 \in \sigma\partial G^*(y^{k+1}) - \sigma K[x^{k+1} + \omega(x^{k+1} - x^k)] + y^{k+1} - y^k, \end{cases} \tag{2.10}$$

where the primal and dual step length parameters $\tau, \sigma > 0$ satisfy $\tau\sigma\|K\| < 1$, and the over-relaxation parameter $\omega = 1$. We can write (2.10) in explicit form as

$$\begin{cases} x^{k+1} := \operatorname{prox}_{\tau F}(x^k - \tau K^* y^k), \\ y^{k+1} := \operatorname{prox}_{\sigma G^*}(y^k + \sigma K[x^{k+1} + \omega(x^{k+1} - x^k)]). \end{cases}$$

2.3 ALGORITHM DERIVATION

The derivation of the PDPS and the optimality conditions (2.8) suggest to solve (2.8) by iteratively solving

$$\begin{cases} B(u^{k+1}, \cdot\,; x^k) = L, \\ B(\cdot\,, w^{k+1}; x^k) = -Q'(u^{k+1}), \\ 0 \in \tau_k \partial F(x^{k+1}) + \tau_k B_x(u^{k+1}, w^{k+1}) + \tau K^* y^k + x^{k+1} - x^k \\ 0 \in \sigma_{k+1} \partial G^*(y^{k+1}) - \sigma_{k+1} K[x^{k+1} + \omega_k(x^{k+1} - x^k)] + y^{k+1} - y^k. \end{cases} \tag{2.11}$$

We have made the step length and over-relaxation parameters iteration-dependent for acceleration purposes. The indexing τ_k and σ_{k+1} is off-by-one to maintain the symmetric update rules from [2].

The method in (2.11) still requires exact solution of the PDEs. For some splitting operators $\Gamma_k, \Upsilon_k : U \times W \times X \to \mathbb{R}$, we therefore transform the first two lines into

$$B(u^{k+1}, \cdot\,; x^k) - \Gamma_k(u^{k+1} - u^k, \cdot\,; x^k) = L \quad \text{and} \tag{2.12a}$$

$$B(\cdot\,, w^{k+1}; x^k) - \Upsilon_k(\cdot\,, w^{k+1} - w^k; x^k) = -Q'(u^{k+1}) \tag{2.12b}$$

Example 2.3 (Splitting). Let $B(u, w; x) = \langle A_x u, w\rangle$ for symmetric $A_x \in \mathbb{R}^{n\times n}$ on $U = W = \mathbb{R}^n$. Take $\Gamma_k(u, w; x) = \langle [A_x - N_x]u, w\rangle$ and $\Upsilon_k = \Gamma_k$ for easily invertible $N_x \in \mathbb{R}^{n\times n}$. With $L = \langle b, \cdot\rangle$ and $M_x := A_x - N_x$, (2.12) now reads

$$N_{x^k} u^{k+1} = b - M_{x^k} u^k \quad \text{and} \quad N_{x^k} w^{k+1} = -\nabla Q(u^{k+1}) - M_{x^k} w^k. \tag{2.13}$$

For Jacobi splitting we take N_{x^k} as the diagonal part of A_{x^k}, and for Gauss–Seidel splitting as the lower triangle including the diagonal. We study these choices further in Section 4.2.

Let us introduce the general notation $v = (u, w, x, y)$ as well as the *step length operators* $T_k \in \mathbb{L}(U^* \times W^* \times X \times Y; U^* \times W^* \times X \times Y)$,

$$T_k := \operatorname{diag}\left(\mathrm{Id}_{U^*} \quad \mathrm{Id}_{W^*} \quad \tau_k \mathrm{Id}_X \quad \sigma_{k+1}\mathrm{Id}_Y\right), \tag{2.14}$$

the set-valued operators $H_k : U \times W \times X \times Y \rightrightarrows U^* \times W^* \times X \times Y$,

$$H_k(v) := \begin{pmatrix} B(u, \cdot\,; x^k) - \Gamma_k(u - u^k, \cdot\,; x^k) - L \\ B(\cdot\,, w; x^k) - \Upsilon_k(\cdot\,, w - w^k; x^k) + Q'(u) \\ \partial F(x) + B_x(u, w) + K^* y \\ \partial G^*(y) - Kx \end{pmatrix}, \tag{2.15}$$

and the *preconditioning operators* $M_k \in \mathbb{L}(U \times W \times X \times Y; U^* \times W^* \times X \times Y)$,

$$M_k := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \mathrm{Id}_X & -\tau_k K^* \\ & & -\omega_k \sigma_{k+1} K & \mathrm{Id}_Y \end{pmatrix}, \tag{2.16}$$

Algorithm 2.1 Primal dual splitting with parallel adaptive PDE solves (PDPAP)

Require: $F : X \to \overline{\mathbb{R}}$, $G^* : Y \to \overline{\mathbb{R}}$, Fréchet-differentiable $Q : U \to \mathbb{R}$; $K \in \mathbb{L}(X; Y)$, $L \in U^*$; and $B : U \times W \times X \to \mathbb{R}$, bilinear in the first two variables, affine in the third, all on Hilbert spaces X, Y, U, and W. Riesz representation $B_x(u, w)$ of $B_{\text{lin}}(u, w; \,\cdot\,)$; see (2.6). For all $k \in \mathbb{N}$, splittings $\Gamma_k, \Upsilon_k : U \times W \times X \to \mathbb{R}$ and step length and over-relaxation parameters $\tau_k, \sigma_{k+1}, \omega_k > 0$; see Theorem 3.10 or 3.11.

Pick an initial iterate $(u^0, w^0, x^0, y^0) \in U \times W \times X \times Y$.

for $k \in \mathbb{N}$ **do**

Solve $u^{k+1} \in U$ from the split weak PDE

$$B(u^{k+1}, \tilde{w}; x^k) - \Gamma_k(u^{k+1} - u^k, \tilde{w}; x^k) = L\tilde{w} \quad \text{for all} \quad \tilde{w} \in W.$$

Solve $w^{k+1} \in W$ from the split weak adjoint PDE

$$B(\tilde{u}, w^{k+1}; x^k) - \Upsilon_k(\tilde{u}, w^{k+1} - w^k; x^k) = -Q'(u^{k+1})\tilde{u} \quad \text{for all} \quad \tilde{u} \in U.$$

$x^{k+1} := \operatorname{prox}_{\tau_k F}\big(x^k - \tau_k B_x(u^{k+1}, w^{k+1}) - \tau_k K^* y^k\big)$

$\bar{x}^{k+1} := x^{k+1} + \omega_k(x^{k+1} - x^k)$

$y^{k+1} := \operatorname{prox}_{\sigma_{k+1} G^*}\big(y^k + \sigma_{k+1} K \bar{x}^{k+1}\big)$

end for

The implicit form of our proposed algorithm for the solution of (2.1) is then

$$0 \in T_k H_k(v^{k+1}) + M_k(v^{k+1} - v^k). \tag{2.17}$$

Writing out (2.17) in terms of explicit proximal maps, we obtain Algorithm 2.1.

Remark 2.4. The index k for T_k, H_k, M_k in (2.14)–(2.17) is inconsistent with some of our earlier articles that would use the index $k+1$ similarly to the unknown v^{k+1}. We have decided to make this change to keep the notation lighter.

3 CONVERGENCE

We now treat the convergence of Algorithm 2.1. Following [30, 7] we “test” its implicit form (2.17) by applying on both sides the linear functional $\langle Z_k \,\cdot\, | v^{k+1} - \bar{v}\rangle$. Here Z_k is a convergence rate encoding “testing operator” (Section 3.2). A simple argument involving the three-point identity (1.2) and a growth estimate for H_k then yields in Section 3.3 a Féjer-type monotonicity estimate in terms of iteration-dependent norms. This establishes in Section 3.4 global convergence subject to a growth condition. We start with assumptions.

3.1 THE MAIN ASSUMPTIONS

We start with our main structural assumption:

Assumption 3.1 (Structure). On Hilbert spaces X, Y, U, and W, we are given convex, proper, and lower semicontinuous $F : X \to \overline{\mathbb{R}}$, $G^* : Y \to \overline{\mathbb{R}}$, and $Q : U \to \mathbb{R}$ with Q Fréchet differentiable, as well as $K \in \mathbb{L}(X; Y)$, $L \in U^*$, and $B : U \times W \times X \to \mathbb{R}$, bilinear in the first two variables, affine in the third. We assume:

(i) F and G are (strongly) convex with factors $\gamma_F, \gamma_{G^*} \geq 0$. With K they satisfy the condition (2.3) for the subdifferential sum and chain rules to be exact.

(ii) For all $x \in X \cap \operatorname{dom} F$, there exist solutions $(u, w) \in U \times W$ to the PDE $B(u, \cdot; x) = L$ and the adjoint PDE $B(\cdot, w; x) = -Q'(u)$.

We then fix a solution $\bar{v} = (\bar{u}, \bar{w}, \bar{x}, \bar{y}) \in U \times W \times X \times Y$ to (2.8) and assume that:

(iii) For some $\mathcal{S}(\bar{u}), \mathcal{S}(\bar{w}) \geq 0$, for all $(u, w) \in U \times W$ and $x \in \operatorname{dom} F$, we have

$$B_{\text{lin}}(u, \bar{w}; x - \bar{x}) \leq \sqrt{\mathcal{S}(\bar{w})}\|u\|_U\|x - \bar{x}\|_X \quad \text{and} \quad B_{\text{lin}}(\bar{u}, w; x - \bar{x}) \leq \sqrt{\mathcal{S}(\bar{u})}\|w\|_W\|x - \bar{x}\|_X.$$

(iv) For some $C_x \geq 0$, for all $(u, w) \in U \times W$ and $x \in \operatorname{dom} F$ we have the bound

$$B_{\text{lin}}(u, w; x - \bar{x}) \leq C_x\|u\|_U\|w\|_W.$$

Remark 3.2. Part (i) is easy to check. In general, (iv) requires $\operatorname{dom} F$ to be bounded with respect to an ∞-norm with $B_{\text{lin}}(u, w, x) \leq C\|u\|_U\|w\|_W\|x\|_\infty$ for some $C > 0$. Then $C_x = \sup_{x \in \operatorname{dom} F} C\|x\|_\infty$. We study (ii)–(iv) further in Section 4.1.

The next assumption introduces *testing parameters* that encode convergence rates and restrict the *step length parameters* in the standard primal-dual component of our method. It has no difference to the treatment of the PDPS in [30, 7]. Dependent on whether both, one, or none of $\tilde{\gamma}_F > 0$ and $\tilde{\gamma}_{G^*} > 0$, the parameters can be chosen to yield varying modes and rates of convergence.

Assumption 3.3 (Primal-dual parameters). Let Assumption 3.1 hold. For all $k \in \mathbb{N}$, the *testing parameters* $\varphi_k, \psi_k > 0$, *step length parameters* $\tau_k, \sigma_k > 0$, and the *over-relaxation parameter* $\omega_k \in (0, 1]$ satisfy for some $\tilde{\gamma}_F \in [0, \gamma_F]$ and $\tilde{\gamma}_{G^*} \in [0, \gamma_{G^*}]$ that

$$\begin{aligned} \varphi_{k+1} &= \varphi_k(1 + 2\tilde{\gamma}_F\tau_k), & \psi_{k+1} &= \psi_k(1 + 2\tilde{\gamma}_{G^*}\sigma_k), & & \\ \eta_k &:= \varphi_k\tau_k = \psi_k\sigma_k, & \omega_k &= \eta_{k+1}^{-1}\eta_k, \quad \text{and} & \kappa &\geq \frac{\tau_k\sigma_k}{1 + 2\tilde{\gamma}_{G^*}\sigma_k}\|K\|^2. \end{aligned}$$

The next assumption encodes our conditions on the PDE splittings.

Assumption 3.4 (Splitting). Let Assumption 3.1 hold. For $k \in \mathbb{N}$, for which this assumption is to hold, we are given splitting operators $\Gamma_k, \Upsilon_k : U \times W \times X \to \mathbb{R}$ and $v^k = (u^k, w^k, x^k, y^k) \in U \times W \times X \times Y$ such that:

(i) Γ_k is linear in the second argument, Υ_k in the first.

(ii) There exist solutions u^{k+1} and w^{k+1} to the split equations (2.12).

(iii) For some $\gamma_B > 0$ and $C_Q, \pi_B \geq 0$, we have

$$\begin{gathered} \|u^k - \bar{u}\|_U^2 \geq \gamma_B\|u^{k+1} - \bar{u}\|_U^2 - \pi_B\|x^k - \bar{x}\|_X^2 \quad \text{and} \\ \|w^k - \bar{w}\|_W^2 \geq \gamma_B\|w^{k+1} - \bar{w}\|_W^2 - C_Q\|u^{k+1} - \bar{u}\|_U^2 - \pi_B\|x^k - \bar{x}\|_X^2. \end{gathered}$$

We verify the assumption for standard splittings in Section 4.2. Generally π_B models the x-sensitivity of B, and will be zero if $B = B_{\text{const}}$. The factor C_Q is related to the Lipschitz factor of Q', and γ_B to the contractivity of the splitting; if $\Upsilon_k = \Gamma_k = 0$, we can take any $\gamma_B > 1$ for appropriate choices of the other factors.

3.2 THE TESTING OPERATOR

To complement the primal-dual testing parameters in Assumption 3.3, we introduce testing parameters $\lambda_k, \theta_k > 0$ corresponding to the PDE updates in our method; the first two lines of (2.17). We combine all of them into the *testing operator* $Z_k \in \mathbb{L}(U^* \times W^* \times X \times Y; U^* \times W^* \times X \times Y)$ defined by

$$Z_k := \operatorname{diag}\left(\lambda_k \mathrm{Id} \quad \theta_k \mathrm{Id} \quad \varphi_k \mathrm{Id} \quad \psi_{k+1} \mathrm{Id}\right). \tag{3.1}$$

Recalling M_k and Z_k from (2.16) and (3.1), thanks to Assumption 3.3, we have

$$Z_k M_k = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \varphi_k \mathrm{Id}_X & -\eta_k K^* \\ & & -\eta_k K & \psi_{k+1} \mathrm{Id}_Y \end{pmatrix}. \tag{3.2}$$

Therefore,

$$Z_k(M_k + \Xi_k) = Z_{k+1} M_{k+1} + D_{k+1} \tag{3.3}$$

for skew-symmetric

$$D_{k+1} := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & (\eta_{k+1} + \eta_k) K^* \\ & & -(\eta_{k+1} + \eta_k) K & 0 \end{pmatrix}$$

and $\Xi_k \in \mathbb{L}(U \times W \times X \times Y; U^* \times W^* \times X \times Y)$ satisfying

$$Z_k \Xi_k = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2\eta_k \tilde{\gamma}_F \mathrm{Id}_X & 2\eta_k K^* \\ & & -2\eta_{k+1} K & 2\eta_{k+1} \tilde{\gamma}_{G^*} \mathrm{Id}_Y \end{pmatrix}. \tag{3.4}$$

Assumption 3.3 ensures $Z_k M_k$ to be positive semi-definite. The proof is exactly as for the PDPS, see, e.g., [7], but we include it for completeness.

Lemma 3.5. *Let $k \in \mathbb{N}$ and suppose Assumption 3.3 holds. Then*

$$Z_k M_k \geq \operatorname{diag}(0, 0, \varphi_k(1-\kappa)\mathrm{Id}_X, \psi_{k+1}\varepsilon \mathrm{Id}_Y) \geq 0 \quad \textit{for} \quad \varepsilon := 1 - \frac{\tau_k \sigma_k}{\kappa(1 + 2\tilde{\gamma}_{G^*} \sigma_k)} \|K\|^2 > 0.$$

Proof. By Young's inequality, for any $v = (u, w, x, y)$,

$$\begin{aligned} \langle Z_k M_k v | v \rangle &= \varphi_k \|x\|_X^2 + \psi_{k+1} \|y\|_Y^2 - 2\eta_k \langle x, K^* y \rangle_X \\ &\geq \varphi_k (1-\kappa) \|x\|_X^2 + \psi_{k+1} \|y\|_Y^2 - \kappa^{-1} \varphi_k \tau_k^2 \|K^* y\|_X^2. \end{aligned}$$

Since $\varphi_k \tau_k^2 = \eta_k \tau_k = \psi_k \sigma_k \tau_k = \psi_{k+1} \sigma_k \tau_k / (1 + 2\tilde{\gamma}_{G^*} \sigma_k)$, the claim follows. □

3.3 GROWTH ESTIMATES AND MONOTONICITY

We start by deriving a three-point monotonicity estimate for H_k. This demands the somewhat strict bounds (3.5).

Lemma 3.6. *Let $k \in \mathbb{N}$. Suppose Assumptions 3.1, 3.3 and 3.4 hold and*

$$\gamma_F \geq \tilde{\gamma}_F + \varepsilon_u + \varepsilon_w + \frac{\lambda_{k+1} + \theta_{k+1}}{\eta_k} \pi_B, \tag{3.5a}$$

$$\gamma_{G^*} \geq \tilde{\gamma}_{G^*}, \tag{3.5b}$$

$$\gamma_B \geq \frac{\lambda_{k+1}}{\lambda_k} + \frac{\theta_k}{\lambda_k} C_Q + \frac{\eta_k \mathcal{S}(\bar{w})}{\varepsilon_w \lambda_k} + \frac{C_x \mu \eta_k}{2\lambda_k}, \quad \textit{and} \tag{3.5c}$$

$$\gamma_B \geq \frac{\theta_{k+1}}{\theta_k} + \frac{\eta_k \mathcal{S}(\bar{u})}{\varepsilon_u \theta_k} + \frac{C_x \eta_k}{2\mu \theta_k} \tag{3.5d}$$

for some $\varepsilon_u, \varepsilon_w, \mu > 0$. Then H_k defined in (2.15) satisfies

$$\begin{aligned} \langle Z_k T_k H_k(v^{k+1}) | v^{k+1} - \bar{v} \rangle \geq{}& \frac{1}{2} \|v^{k+1} - \bar{v}\|^2_{Z_k \Xi_k} \\ &+ (\lambda_{k+1} + \theta_{k+1}) \pi_B \|x^{k+1} - \bar{x}\|^2_X - (\lambda_k + \theta_k) \pi_B \|x^k - \bar{x}\|^2_X \\ &+ \lambda_{k+1} \|u^{k+1} - \bar{u}\|^2_U - \lambda_k \|u^k - \bar{u}\|^2_U \\ &+ \theta_{k+1} \|w^{k+1} - \bar{w}\|^2_W - \theta_k \|w^k - \bar{w}\|^2_W. \end{aligned} \tag{3.6}$$

Proof. For brevity we denote $v = (u, w, x, y) = v^{k+1}$. Recall that $\bar{v} = (\bar{u}, \bar{w}, \bar{x}, \bar{y})$ satisfies by Assumption 3.1 the optimality conditions (2.8). Since Algorithm 2.1 guarantees the first two lines of H_k to be zero through the choice of M_k in (2.16), introducing $q_F := -B_x(\bar{u}, \bar{w}) - K^* \bar{y} \in \partial F(\bar{x})$ we expand

$$\begin{aligned} \langle Z_k T_k H_k(v) | v - \bar{v} \rangle &= \eta_k \langle \partial F(x) + B_x(u, w) + K^* y, x - \bar{x} \rangle_X + \eta_{k+1} \langle \partial G^*(y) - Kx, y - \bar{y} \rangle_Y \\ &= \eta_k \langle \partial F(x) - q_F, x - \bar{x} \rangle + \eta_k \langle B_x(u, w) - B_x(\bar{u}, \bar{w}), x - \bar{x} \rangle \\ &\quad + \eta_{k+1} \langle \partial G^*(y) - K\bar{x}, y - \bar{y} \rangle + (\eta_k - \eta_{k+1}) \langle K(x - \bar{x}), y - \bar{y} \rangle. \end{aligned}$$

Using (3.4) we also have

$$\frac{1}{2} \|v - \bar{v}\|^2_{Z_k \Xi_k} = \eta_k \tilde{\gamma}_F \|x - \bar{x}\|^2_X + (\eta_k - \eta_{k+1}) \langle K(x - \bar{x}), y - \bar{y} \rangle_Y + \eta_{k+1} \tilde{\gamma}_{G^*} \|\|y - \bar{y}\|^2_Y.$$

We now use the (strong) monotonicity of F and G^* with constants γ_F and γ_{G^*} contained Assumption 3.1 (i), as well as the splitting inequality Assumption 3.4 (iii). Thus

$$\begin{aligned} \langle Z_k T_k H_k(v) | v - \bar{v} \rangle \geq{}& \frac{1}{2} \|v - \bar{v}\|^2_{Z_k \Xi_k} + \eta_k (\gamma_F - \tilde{\gamma}_F) \|x - \bar{x}\|^2_X - (\lambda_k + \theta_k) \pi_B \|x^k - \bar{x}\|^2_X \\ &+ \eta_{k+1} (\gamma_{G^*} - \tilde{\gamma}_{G^*}) \|y - \bar{y}\|^2_Y + \eta_k \langle B_x(u, w) - B_x(\bar{u}, \bar{w}), x - \bar{x} \rangle \\ &+ (\lambda_k \gamma_B - \theta_k C_Q) \|u - \bar{u}\|^2_U - \lambda_k \|u^k - \bar{u}\|^2_U \\ &+ \theta_k \gamma_B \|w - \bar{w}\|^2_W - \theta_k \|w^k - \bar{w}\|^2_W. \end{aligned} \tag{3.7}$$

The Riesz equivalence (2.6), Assumption 3.1 (iii) and (iv), and Young's inequality give

$$\begin{aligned} \eta_k \langle B_x(u, w) - B_x(\bar{u}, \bar{w}), x - \bar{x} \rangle &= \eta_k B_{\text{lin}}(u - \bar{u}, w - \bar{w}; x - \bar{x}) + \eta_k B_{\text{lin}}(\bar{u}, w - \bar{w}; x - \bar{x}) \\ &\quad - \eta_k B_{\text{lin}}(\bar{u} - u, \bar{w}; x - \bar{x}) \\ &\geq -\eta_k \left(\frac{\mathcal{S}(\bar{u})}{\varepsilon_u} + \frac{C_x \mu}{2} \right) \|w - \bar{w}\|^2_W - \eta_k \left(\frac{\mathcal{S}(\bar{w})}{\varepsilon_w} + \frac{C_x}{2\mu} \right) \|u - \bar{u}\|^2_U \\ &\quad - \eta_k (\varepsilon_u + \varepsilon_w) \|x - \bar{x}\|^2_X. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\begin{aligned}\langle Z_k T_k H_k(v)|v-\bar v\rangle \geq \frac{1}{2}\|v-\bar v\|^2_{Z_k\Xi_k} + \eta_{k+1}(\gamma_{G^*}-\tilde\gamma_{G^*})\|y-\bar y\|^2_Y \\ + \eta_k(\gamma_F-\tilde\gamma_F-\varepsilon_u-\varepsilon_w)\|x-\bar x\|^2_X - (\lambda_k+\theta_k)\pi_B\|x^k-\bar x\|^2_X \\ - \lambda_k\|u^k-\bar u\|^2_U + \lambda_k\left(\gamma_B - \frac{\theta_k}{\lambda_k}C_Q - \frac{\eta_k\mathcal{S}(\bar w)}{\varepsilon_w\lambda_k} - \frac{C_x\mu\eta_k}{2\lambda_k}\right)\|u-\bar u\|^2_U \\ - \theta_k\|w^k-\bar w\|^2_W + \theta_k\left(\gamma_B - \frac{\eta_k\mathcal{S}(\bar u)}{\varepsilon_u\theta_k} - \frac{C_x\eta_k}{2\mu\theta_k}\right)\|w-\bar w\|^2_W.\end{aligned}$$

The claim now follows by applying (3.5). □

We now simplify and interpret (3.5).

Lemma 3.7. *Let $k\in\mathbb{N}$. Suppose $\gamma_F>\tilde\gamma_F>0$ as well as $\gamma_{G^*}\geq\tilde\gamma_{G^*}\geq 0$ and that there exists $c>0$ with $\eta_{k+1}\leq c\eta_k$ such that*

$$\gamma_B \geq c + \frac{\mathcal{S}(\bar u)}{\mathcal{S}(\bar w)}C_Q + \frac{2c\pi_B(\mathcal{S}(\bar u)+\mathcal{S}(\bar w))}{(\gamma_F-\tilde\gamma_F)^2}\left(4+\frac{C_x(\gamma_F-\tilde\gamma_F)}{2\sqrt{\mathcal{S}(\bar w)\mathcal{S}(\bar u)}}\right). \tag{3.9}$$

Then there exist $\varepsilon_u,\varepsilon_w,\mu>0$ such that (3.5) holds for

$$\lambda_k = a\mathcal{S}(\bar w)\eta_k \quad\text{and}\quad \theta_k = a\mathcal{S}(\bar u)\eta_k,\quad\text{where}\quad a := \frac{\gamma_F-\tilde\gamma_F}{2c\pi_B(\mathcal{S}(\bar u)+\mathcal{S}(\bar w))}. \tag{3.10}$$

Proof. Take $\varepsilon_u=\varepsilon_w=(\gamma_F-\tilde\gamma_F)/4$, and $\mu=\sqrt{\lambda_k/\theta_k}=\sqrt{\mathcal{S}(\bar w)/\mathcal{S}(\bar u)}$. It follows

$$\frac{1}{a\sqrt{\mathcal{S}(\bar w)\mathcal{S}(\bar u)}} = \frac{\eta_k}{\sqrt{\lambda_k\theta_k}} = \frac{\mu\eta_k}{\lambda_k} = \frac{\eta_k}{\mu\theta_k},$$

as well as $\gamma_F = \tilde\gamma_F+\varepsilon_u+\varepsilon_w+c\pi_B a(\mathcal{S}(\bar u)+\mathcal{S}(\bar w))$. Therefore

$$\frac{2c\pi_B(\mathcal{S}(\bar u)+\mathcal{S}(\bar w))}{(\gamma_F-\tilde\gamma_F)^2} = \frac{1}{(\gamma_F-\tilde\gamma_F)a} = \frac{1}{4\varepsilon_u a} = \frac{1}{4\varepsilon_w a}.$$

Then (3.9) implies

$$\gamma_B \geq c + \frac{\mathcal{S}(\bar u)}{\mathcal{S}(\bar w)}C_Q + \frac{1}{\varepsilon_w a} + \frac{C_x\mu\eta_k}{2\lambda_k} \quad\text{and}\quad \gamma_B \geq c + \frac{1}{\varepsilon_u a} + \frac{C_x\eta_k}{2\mu\theta_k}.$$

Using $\lambda_k=a\mathcal{S}(\bar w)\eta_k$ and $\theta_k=a\mathcal{S}(\bar u)\eta_k$ as well as $c\geq\eta_{k+1}/\eta_k=\lambda_{k+1}/\lambda_k=\theta_{k+1}/\theta_k$, these two inequalities, and $\gamma_{G^*}\geq\tilde\gamma_{G^*}$ yield (3.5). □

Remark 3.8. Since $\eta_{k+1}\geq\eta_k$ for convergent algorithms, i.e., $c\geq 1$, letting $c=1$ and $\tilde\gamma_F=0$ in (3.9), we obtain at the solution $(\bar u,\bar w,\bar x,\bar y)$ a fundamental "second order growth" and splitting condition (via C_Q and π_B) that cannot be avoided by step length parameter choices.

Our convergence proof is based based on the next Féjer-type monotonicity estimate with respect to the iteration-dependent norms $\|\cdot\|_{Z_k\tilde M_k}$. Denoting by $\mathrm{In}_U: U\hookrightarrow U^*$ the canonical injection, $\langle \mathrm{In}u|\tilde u\rangle = \langle u,\tilde u\rangle$ for all $u,\tilde u\in U$, here $\tilde M_k\in\mathbb{L}(U\times W\times X\times Y; U^*\times W^*\times X\times Y)$ modifies M_k defined in (2.16) as

$$\tilde M_k := M_k + \operatorname{diag}\begin{pmatrix}\mathrm{In}_U & \mathrm{In}_W & \varphi_k^{-1}(\lambda_k+\theta_k)\pi_B\mathrm{Id}_X & 0\end{pmatrix}.$$

By (3.2) and Assumption 3.3, this satisfies

$$Z_k\tilde M_k = \begin{pmatrix}\lambda_k\mathrm{In}_U & & & \\ & \theta_k\mathrm{In}_W & & \\ & & (\varphi_k+(\lambda_k+\theta_k)\pi_B)\mathrm{Id}_X & -\eta_kK^* \\ & & -\eta_kK & \psi_{k+1}\mathrm{Id}_Y\end{pmatrix}. \tag{3.11}$$

Lemma 3.9. *Suppose Assumptions 3.1 and 3.3 hold as does Assumption 3.4 and (3.5) for* $k = 0, \ldots, N$. *Given* v^0, *let* $v^1, \ldots, v^{N-1}$ *be produced by Algorithm 2.1. Then*

$$\frac{1}{2}\|v^{k+1} - \bar{v}\|^2_{Z_{k+1}\tilde{M}_{k+1}} + \frac{1}{2}\|v^{k+1} - v^k\|^2_{Z_k M_k} \le \frac{1}{2}\|v^k - \bar{v}\|^2_{Z_k \tilde{M}_k} \quad (k = 0, \ldots, N-1) \tag{3.12}$$

where all the terms are non-negative.

Proof. Lemma 3.6 gives the estimate

$$\begin{aligned} \langle Z_k T_k H_k(v^{k+1}) | v^{k+1} - \bar{v}\rangle \ge{}& \frac{1}{2}\|v^{k+1} - \bar{v}\|^2_{Z_k \Xi_k} \\ &+ (\lambda_{k+1} + \theta_{k+1})\pi_B\|x^{k+1} - \bar{x}\|^2_X - (\lambda_k + \theta_k)\pi_B\|x^k - \bar{x}\|^2_X \\ &+ \lambda_{k+1}\|u^{k+1} - \bar{u}\|^2_U - \lambda_k\|u^k - \bar{u}\|^2_U \\ &+ \theta_{k+1}\|w^{k+1} - \bar{w}\|^2_W - \theta_k\|w^k - \bar{w}\|^2_W \\ ={}& \frac{1}{2}\|v^{k+1} - \bar{v}\|^2_{Z_{k+1}(\tilde{M}_{k+1} - M_{k+1}) + Z_k\Xi_k} - \frac{1}{2}\|v^k - \bar{v}\|^2_{Z_k(\tilde{M}_k - M_k)}. \end{aligned} \tag{3.13}$$

By the implicit form (2.17) of Algorithm 2.1, we have $-Z_k M_k(v^{k+1} - v^k) \in Z_k T_k H_k(v^{k+1})$. Thus (3.13) combined with the three-point identity (1.2) for the operator $M = Z_k M_k$ yields

$$\frac{1}{2}\|v^k - \bar{v}\|^2_{Z_k\tilde{M}_k} \ge \frac{1}{2}\|v^{k+1} - \bar{v}\|^2_{Z_{k+1}(\tilde{M}_{k+1} - M_{k+1}) + Z_k(M_k + \Xi_k)} + \frac{1}{2}\|v^{k+1} - v^k\|^2_{Z_k M_k}$$

Therefore (3.12) follows by applying (3.3), i.e., $Z_k(M_k + \Xi_k) = Z_{k+1}M_{k+1} + D_k$, where the skew symmetric term D_k does not contribute to the norms. Finally, we have $Z_k\tilde{M}_k \ge Z_k M_k \ge 0$ by Lemma 3.5, proving the non-negativity of all the terms. □

3.4 MAIN RESULTS

We can now state our main convergence theorems. In terms of assumptions, the only fundamental difference between the accelerated $O(1/N)$ and the linear convergence result is that the latter requires G^* to be strongly convex and the former doesn't. Both require sufficient second order growth in terms of the respective technical conditions (3.14b) or (3.17b). The step length parameters differ.

Theorem 3.10 (Accelerated convergence). *Suppose Assumptions 3.1 and 3.4 hold with* $\gamma_F > 0$. *Pick* $\tau_0, \sigma_0, \kappa > 0$, $0 < \tilde{\gamma}_F < \gamma_F$, *and* $\tilde{\gamma}_{G^*} = 0$ *satisfying*

$$1 > \kappa \ge \tau_0\sigma_0\|K\|^2 \quad \text{and} \tag{3.14a}$$

$$\gamma_B \ge \omega_0^{-1} + \frac{\mathcal{S}(\bar{u})}{\mathcal{S}(\bar{w})}C_Q + \frac{2\pi_B(\mathcal{S}(\bar{u}) + \mathcal{S}(\bar{w}))}{\omega_0(\gamma_F - \tilde{\gamma}_F)^2}\left(4 + \frac{C_x(\gamma_F - \tilde{\gamma}_F)}{2\sqrt{\mathcal{S}(\bar{w})\mathcal{S}(\bar{u})}}\right), \tag{3.14b}$$

where ω_0 *is defined as part of the update rules*

$$\tau_{k+1} := \tau_k\omega_k, \quad \sigma_{k+1} := \sigma_k/\omega_k, \quad \text{and} \quad \omega_k := 1/\sqrt{1 + 2\tilde{\gamma}_F\tau_k} \quad (k \in \mathbb{N}).$$

Let $\{v^{k+1}\}_{k\in\mathbb{N}}$ *be generated by Algorithm 2.1 for any* $v^0 \in U \times W \times X \times Y$. *Then* $x^k \to \bar{x}$ *in* X; $u^k \to \bar{u}$ *in* U; *and* $w^k \to \bar{w}$ *in* W, *all strongly at the rate* $O(1/N)$.

Proof. We use Lemma 3.9, whose assumptions we now verify. Assumptions 3.1 and 3.4 we have assumed. As shown in [30, 7], Assumption 3.3 holds with $\psi_k \equiv \sigma_0^{-1}\tau_0$, $\varphi_0 = 1$, and $\varphi_{k+1} := \varphi_k/\omega_k^2$. Moreover, $\{\varphi_k\}_{k\in\mathbb{N}}$ grows at the rate $\Omega(k^2)$. Hence

$$\eta_{k+1} = \omega_k^{-1}\eta_k = \sqrt{1 + 2\tilde{\gamma}_F\tau_k}\,\eta_k \le c\eta_k \quad \text{for} \quad c = \omega_0^{-1} = \sqrt{1 + 2\tilde{\gamma}_F\tau_0}.$$

Thus (3.14) verifies (3.9) so that Lemma 3.7 verifies (3.5). Thus we may apply Lemma 3.9. By summing its result over $k = 0, \ldots, N-1$, we get

$$\frac{1}{2}\|v^N - \bar{v}\|^2_{Z_N \tilde{M}_N} \leq \frac{1}{2}\|v^0 - \bar{v}\|^2_{Z_0 \tilde{M}_0}. \tag{3.15}$$

By (3.2), (3.11), and Lemma 3.5 we have

$$Z_k \tilde{M}_k \geq Z_k M_k \geq \operatorname{diag}\left(\lambda_k \mathrm{In}_U \quad \theta_k \mathrm{In}_W \quad \varphi_k(1-\kappa)\mathrm{Id}_X \quad \psi_{k+1}\varepsilon \mathrm{Id}_Y\right) \geq 0. \tag{3.16}$$

where $\varepsilon := 1 - \tau_k\sigma_k\kappa^{-1}\|K\|^2 = 1 - \tau_0\sigma_0\kappa^{-1}\|K\|^2 > 0$ by assumption. By Lemma 3.7, $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{\theta_k\}_{k\in\mathbb{N}}$ grow at the same $\Omega(k^2)$ rate as $\{\varphi_k\}_{k\in\mathbb{N}}$. Therefore (3.15) and (3.16) establish $\|x^k - \bar{x}\|^2_X \to 0$ as well as $\|u^k - \bar{u}\|^2_U$ and $\|w^k - \bar{w}\|^2_W \to 0$, all at the rate $O(1/N^2)$. The claim follows by removing the squares. ◻

Theorem 3.11 (Linear convergence). *Suppose Assumptions 3.1 and 3.4 hold with both $\gamma_F > 0$ and $\gamma_{G^*} > 0$. Pick $\tau, \kappa > 0$, $0 < \tilde{\gamma}_F \leq \gamma_F$, $0 < \tilde{\gamma}_{G^*} \leq \gamma_{G^*}$ satisfying*

$$1 > \kappa \geq \tau^2 \tilde{\gamma}_{G^*}^{-1}\tilde{\gamma}_F\|K\|^2 \quad \text{and} \tag{3.17a}$$

$$\gamma_B \geq \omega^{-1} + \frac{\mathcal{S}(\bar{u})}{\mathcal{S}(\bar{w})}C_Q + \frac{2\pi_B\left(\mathcal{S}(\bar{u}) + \mathcal{S}(\bar{w})\right)}{\omega(\gamma_F - \tilde{\gamma}_F)^2}\left(4 + \frac{C_x(\gamma_F - \tilde{\gamma}_F)}{2\sqrt{\mathcal{S}(\bar{w})\mathcal{S}(\bar{u})}}\right) \tag{3.17b}$$

for

$$\sigma := \tilde{\gamma}_{G^*}^{-1}\tilde{\gamma}_F\tau \quad \text{and} \quad \omega := 1/(1 + 2\tilde{\gamma}_F\tau) = 1/(1 + \tilde{\gamma}_{G^*}\sigma).$$

Take $\tau_k \equiv \tau$, $\sigma_k \equiv \sigma$, and $\omega_k \equiv \omega$. Let $\{v^{k+1}\}_{k\in\mathbb{N}}$ be generated by Algorithm 2.1 for any $v^0 \in U \times W \times X \times Y$. Then $x^k \to \bar{x}$ in X; $u^k \to \bar{u}$ in U; and $w^k \to \bar{w}$ in W, all strongly at a linear rate.

Proof. As shown in [30, 7], Assumption 3.3 is satisfied for $\varphi_0 = 1$, $\psi_0 = \sigma^{-1}\tau$, $\varphi_{k+1} := \varphi_k/\omega_k$, and $\psi_{k+1} := \psi_k/\omega_k$. Moreover, both $\{\varphi_k\}_{k\in\mathbb{N}}$ and $\{\psi_k\}_{k\in\mathbb{N}}$ grow exponentially and $\eta_{k+1} \leq \omega^{-1}\eta_k$. Thus (3.17) verifies (3.9) with $c = \omega^{-1}$ so that Lemma 3.7 verifies (3.5). The rest follows as in the proof of Theorem 3.10. ◻

Theorems 3.10 and 3.11 show global convergence, but may require a very constricted dom F through the constant C_x in Assumption 3.1 (iv). In Appendix A we relax the constant by localizing the convergence.

Remark 3.12 (Weak convergence). It is possible to prove weak convergence when $\omega \equiv 1$ and $\tau \equiv \tau_0$, $\sigma \equiv \sigma_0$ satisfy (3.14). The proof is based on an extension of Opial's lemma to the quantitative Féjer monotonicity (3.12). We have not included the proof since it is technical, and does not permit reducing assumptions from those of Theorems 3.10 and 3.11. We refer to [4] for the corresponding proof for the NL-PDPS.

4 SPLITTINGS AND PARTIAL DIFFERENTIAL EQUATIONS

We now prove Assumption 3.1 and derive explicit expressions for the operator B_x from (2.6). We do this in Section 4.1 for some sample PDEs. Then in Section 4.2 we study the satisfaction of Assumption 3.4 for Gauss–Seidel and Jacobi splitting. We briefly discuss a quasi-conjugate gradient splitting to illustrate the generality of our approach.

4.1 PARTIAL DIFFERENTIAL EQUATIONS AND RIESZ REPRESENTATIONS

Let $\mathrm{Sym}^d \subset \mathbb{R}^{d\times d}$ stand for the symmetric matrices. With $\Omega \subset \mathbb{R}^d$ a Lipschitz domain, we take

$$x = (A, c) \in X := X_1 \times X_2 \quad \text{for subspaces} \quad X_1 \subset L^2(\Omega; \mathrm{Sym}^d) \quad \text{and} \quad X_2 \subset L^2(\Omega). \tag{4.1a}$$

Then with $U = H^1(\Omega)$ and $W = H^1_0(\Omega) \times H^{1/2}(\partial\Omega)$, we define

$$B(u, w; x) := B_{\mathrm{lin}}(u, w; x) + B_{\mathrm{const}}(u, w) \quad \text{for} \quad u \in U,\ w \in W,\ x \in X. \tag{4.1b}$$

where, writing $w = (w_\Omega, w_\partial)$,

$$B_{\mathrm{lin}}(u, w; x) := \langle \nabla u, A\nabla w_\Omega\rangle_{L^2(\Omega)} + \langle cu, w_\Omega\rangle_{L^2(\Omega)} \quad \text{and} \tag{4.1c}$$

$$B_{\mathrm{const}}(u, w) := \langle \mathrm{trace}_{\partial\Omega}\, u, w_\partial\rangle_{L^2(\partial\Omega)}. \tag{4.1d}$$

Thus B_{const} models the nonhomogeneous Dirichlet boundary condition $u = g$ on $\partial\Omega$ for some $g \in H^{-\frac{1}{2}}(\partial\Omega)$. Correspondingly we take for some $L_0 \in U^*$ as the right-hand-side

$$Lw := L_0 w_\Omega + \langle g, w_\partial\rangle_{L^2(\partial\Omega)}. \tag{4.1e}$$

The next lemma verifies the PDE components of Assumption 3.1. Afterwards we look at particular choices of X_1 and X_2. We could also take $W = H^1(\Omega)$, $w = w_\Omega$, $L = L_0$, and $B_{\mathrm{const}} = 0$ to model Neumann boundary conditions, and the result would still hold. For the norms in $L^p(\Omega; \mathbb{R}^d)$, $W^{1,p}(\Omega)$, and $L^p(\Omega; \mathbb{R}^{d\times d})$, we use the Euclidean norm in $\mathbb{R}^d$ and the spectral norm $\|\cdot\|_2$ in $\mathbb{R}^{d\times d}$.

Lemma 4.1. *Assume* (4.1) *and that* $\mathrm{dom}\, F \subset L^\infty(\Omega; \mathbb{R}^{d\times d}) \times L^\infty(\Omega)$. *Then:*

(ii′) Assumption 3.1 (ii) holds if there exists $\lambda \in (0, 1)$ *such that*

$$A(\xi) \geq \lambda\, \mathrm{Id} \quad \textit{and} \quad |c(\xi)| \geq \lambda \quad \textit{for all} \quad \xi \in \Omega \quad \textit{and} \quad (A, c) \in (X_1 \times X_2) \cap \mathrm{dom}\, F.$$

If $\bar v = (\bar u, \bar w, \bar x, \bar y)$ *solves* (2.8) *with* $\bar x = (\bar A, \bar c) \in (X_1 \times X_2) \cap \mathrm{dom}\, F$, $\bar u \in H^1(\Omega) \cap W^{1,\infty}(\Omega)$, $\bar w = (\bar w_\Omega, \bar w_\partial) \in (H^1_0(\Omega) \cap W^{1,\infty}(\Omega)) \times H^{1/2}(\partial\Omega)$, *and* $\bar y \in Y$ *for a Hilbert space* Y, *also:*

(iii′) Assumption 3.1 (iii) holds with $\mathcal{S}(\bar u) = \|\bar u\|^2_{W^{1,\infty}(\Omega)}$ *and* $\mathcal{S}(\bar w) = \|\bar w_\Omega\|^2_{W^{1,\infty}(\Omega)}$.

(iv′) Assumption 3.1 (iv) holds with

$$C_x = \sup_{(A,c)\in \mathrm{dom}\, F} \|A - \bar A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} + \|c - \bar c\|_{L^\infty(\Omega)}.$$

Proof. For (ii′), we identify $g \in H^{-1/2}(\partial\Omega)$ with $\hat g \in H^{1/2}(\partial\Omega)$ by the Riesz mapping and fix $\hat u \in H^1(\Omega)$ with $\mathrm{trace}_{\partial\Omega}\, \hat u = \hat g$. This is possible by the definition of $H^{1/2}(\partial\Omega)$. By the Lax–Milgram lemma there is then a unique solution $v \in H^1_0(\Omega)$ to

$$\langle \nabla v, A\nabla w_\Omega\rangle_{L^2(\Omega)} + \langle cv, w_\Omega\rangle_{L^2(\Omega)} = L_0 w_\Omega - B_{\mathrm{lin}}(\hat u, w_\Omega; x) \quad \text{for all} \quad w_\Omega \in H^1_0(\Omega),$$

Now $u = v + \hat u$ satisfies $B(u, w; x) = Lw$ and is independent of the choice of $\hat u$. Analogously we prove the existence of a solution to the adjoint equation.

To prove (iv′), pick arbitrary $u \in H^1(\Omega)$, $w = (w_\Omega, w_\partial) \in H^1_0(\Omega) \times H^{1/2}(\partial\Omega)$, and $x = (A, c) \in (X_1 \times X_2) \cap \mathrm{dom}\, F$. Hölder's inequality and the symmetry of $A(\xi)$ give

$$\begin{aligned}\langle \nabla u, A\nabla w_\Omega\rangle_{L^2(\Omega)} &\leq \|\nabla w_\Omega\|_{L^2(\Omega;\mathbb{R}^d)} \left(\int_\Omega \|A(\xi)\nabla u(\xi)\|_2^2\, d\xi\right)^{1/2} \\ &\leq \|\nabla w_\Omega\|_{L^2(\Omega;\mathbb{R}^d)} \|A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, as claimed

$$\begin{aligned} B_{\text{lin}}(u, w; x - \bar{x}) &\leq \|A - \bar{A}\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla w_\Omega\|_{L^2(\Omega;\mathbb{R}^d)} \\ &\quad + \|c - \bar{c}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|w_\Omega\|_{L^2(\Omega)} \\ &\leq \left(\|A - \bar{A}\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} + \|c - \bar{c}\|_{L^\infty(\Omega)}\right) \|u\|_{H^1(\Omega)} \|w_\Omega\|_{H^1(\Omega)} \\ &\leq C_x \|u\|_{H^1(\Omega)} \|w_\Omega\|_{H^1(\Omega)}. \end{aligned}$$

For (iii′), using Hölder's twice inequality and the symmetry of $A(\xi)$, we estimate

$$\begin{aligned} \langle \nabla u, A\nabla w_\Omega \rangle_{L^2(\Omega)} &\leq \|\nabla w_\Omega\|_{L^\infty(\Omega;\mathbb{R}^d)} \int_\Omega \|A(\xi)\nabla u(\xi)\|_2 \, d\xi \\ &\leq \|\nabla w_\Omega\|_{L^\infty(\Omega;\mathbb{R}^d)} \|A\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Hence

$$\begin{aligned} B_{\text{lin}}(u, \bar{w}; x) &\leq \|A\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla \bar{w}_\Omega\|_{L^\infty(\Omega;\mathbb{R}^d)} \\ &\quad + \|c\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \|\bar{w}_\Omega\|_{L^\infty(\Omega)} \\ &\leq \left(\|\nabla \bar{w}_\Omega\|_{L^\infty(\Omega;\mathbb{R}^d)} + \|\bar{w}_\Omega\|_{L^\infty(\Omega)}\right) \|u\|_{H^1(\Omega)} \left(\|A\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|c\|_{L^2}\right) \\ &= \|\bar{w}_\Omega\|_{W^{1,\infty}(\Omega)} \|u\|_{H^1(\Omega)} \|x\|_X. \end{aligned}$$

Thus we may take as claimed $\mathcal{S}(\bar{w}) = \|\bar{w}_\Omega\|^2_{W^{1,\infty}(\Omega)}$, and analogously $\mathcal{S}(\bar{u}) = \|\bar{u}\|^2_{W^{1,\infty}(\Omega)}$. □

To describe B_x we denote the double dot product and the outer product by

$$A : \tilde{A} = \sum_{ij} A_{ij}\tilde{A}_{ij}, \quad \text{and} \quad v \otimes w = vw^T \quad \text{for} \quad A, \tilde{A} \in \mathbb{R}^{d\times d} \quad \text{and} \quad v, w \in \mathbb{R}^d$$

Observe the identity $v^T A w = A : (v \otimes w)$.

Example 4.2 (General case). In the fully general case, formally and without regard for the solvability of the PDE (2.2), we equip $X_1 = L^2(\Omega; \mathbb{R}^{d\times d})$ with the inner product $\langle A_1, A_2 \rangle_{X_1} := \int_\Omega A_1(\xi) : A_2(\xi) \, d\xi$ and $X_2 = L^2(\Omega; \mathbb{R})$ with the standard inner product in $L^2(\Omega; \mathbb{R})$. Then for all $u \in U$, $w \in W$, and $(d, h) \in X_1 \times X_2$, we have

$$B_{\text{lin}}(u, w; (d, h)) = \langle \nabla u, d\nabla w \rangle_{L^2} + \langle hu, w \rangle_{L^2} = \langle \nabla u \otimes \nabla w, d \rangle_{X_1} + \langle uw, h \rangle_{X_2}.$$

Therefore the Riesz representation B_x has pointwise in Ω the expression

$$B_x(u, w) = \begin{pmatrix} \nabla u \otimes \nabla w \\ uw \end{pmatrix}.$$

The constant C_x is as provided by Lemma 4.1.

Example 4.3 (Scalar function diffusion coefficient). Let then $X_1 := \{\xi \mapsto a(\xi)\text{Id} \mid a \in L^2(\Omega)\}$. X_1 is isometrically isomorphic with $L^2(\Omega)$ since the spectral norm $\|a(\xi)\text{Id}\|_2 = |a(\xi)|$. We may therefore identify X_1 and $L^2(\Omega)$. We also observe that the term $\langle \nabla u, A\nabla w \rangle_{L^2} = \langle a, \nabla u \cdot \nabla w \rangle_{X_1}$. Hence, pointwise in Ω,

$$B_x(u, w) = \begin{pmatrix} \nabla u \cdot \nabla w \\ uw \end{pmatrix}.$$

According to Lemma 4.1, the constant

$$C_x = \sup_{(a,c)\in \operatorname{dom} F} \|a-\bar a\|_{L^\infty(\Omega)} + \|c-\bar c\|_{L^\infty(\Omega)}.$$

Example 4.4 (Spatially uniform coefficients). Let $X_1 := \{\xi \mapsto \tilde A \mid \tilde A \in \mathrm{Sym}^d\} \subset L^2(\Omega; \mathrm{Sym}^d)$ and $X_2 := \{\xi \mapsto \tilde c \mid \tilde c \in \mathbb{R}\} \subset L^2(\Omega)$ consist of constant functions $A : \xi \mapsto \tilde A$ and $c : \xi \mapsto \tilde c$ on the bounded domain Ω. Then $\|x\|_{X_1\times X_2} = |\Omega|^{1/2}(\|\tilde A\|_2 + |\tilde c|)$ for all $x = (A, c) \in X_1 \times X_2$. We may thus identify X_1 and X_2 with $\mathbb{R}^{d\times d}$ and $\mathbb{R}$ if we weigh the norms by $|\Omega|^{1/2}$. We have

$$\langle \nabla u, A\nabla w\rangle_{L^2} = \int_\Omega \tilde A : \nabla u \otimes \nabla w \, d\xi = \tilde A : \int_\Omega \nabla u \otimes \nabla w \, d\xi.$$

Thus

$$B_x(u,w) = \begin{pmatrix} \int_\Omega \nabla u \otimes \nabla w \, d\xi \\ \int_\Omega uw \, d\xi \end{pmatrix}.$$

According to Lemma 4.1, the constant

$$C_x = \sup_{(A,c)\in \operatorname{dom} F} \|\tilde A - \tilde{\bar A}\|_2 + |\tilde c - \tilde{\bar c}|.$$

4.2 SPLITTINGS

We now discuss linear system splittings and Assumption 3.4. Throughout this subsection we concentrate on a discretised version of (4.1b), that is

$$B(u,w;x) = \langle A_x u, w\rangle \quad \text{and} \quad Lw = \langle b, w\rangle \quad \text{on } U = \mathbb{R}^n \tag{4.2}$$

with $A_x \in \mathbb{R}^{n\times n}$ invertible for $x \in X$, and $b \in \mathbb{R}^n$. Then for fixed $x \in X$ the weak PDE (2.2) and the adjoint $B(\,\cdot\,, w, x) = -Q'(u)$ reduce the matrix linear equations

$$A_x u = b \quad \text{and} \quad A_x w = -\nabla Q(u).$$

The basic splittings The next lemma helps to prove Assumption 3.4 subject to a control on the rate of dependence of A on x. In its setting, with $A_x = N_x + M_x$ with N_x "easily" invertible, Lines 3 and 4 of Algorithm 2.1 are given by (2.13).

Theorem 4.5. *In the setting* (4.2)*, suppose Assumption 3.1 holds and*

$$\|A_x - A_{\tilde x}\| \le L_A \|x-\tilde x\| \quad (x, \tilde x \in \operatorname{dom} F) \tag{4.3}$$

for some $L_A \ge 0$. Split $A_x = N_x + M_x$ with N_x invertible, and assume there exist $\alpha \in [0,1)$ and $\gamma_N > 0$ such that both the spectral radii

$$\rho(N_x^{-1}M_x) \le \alpha \quad \textit{and} \quad N_x^* N_x \ge \gamma_N^2 \mathrm{Id} \quad (x \in \operatorname{dom} F). \tag{4.4}$$

Also suppose ∇Q is L_Q-Lipschitz. For any $\gamma_B \in (1, 1/\alpha^2)$, $\lambda \in (0,1)$, and $\beta > 0$, set

$$\pi_B = \left(1 + \beta + \frac{\alpha^2\gamma_B}{\lambda(1-\alpha^2\gamma_B)}\right)\frac{\gamma_B L_A^2 \|\bar w\|^2}{\gamma_N^2} \quad \textit{and} \quad C_Q = \left(\frac{1+\beta}{\beta} + \frac{\alpha^2\gamma_B}{(1-\lambda)(1-\alpha^2\gamma_B)}\right)\frac{\gamma_B L_Q^2}{\gamma_N^2}.$$

Let $\Gamma_k(u,w,x) = \langle M_x u, w\rangle$ and $\Upsilon_k(u,w,x) = \langle u, M_x w\rangle$. Then Assumption 3.4 holds for all $k \in \mathbb{N}$ with $\{v^{k+1}\}_{k=0}^\infty$ generated by Algorithm 2.1 for any $v^0 \in U \times W \times X \times Y$.

Proof. Assumption 3.4 (i) holds by construction, and (ii) by the assumed invertibility of N_x for $x \in \operatorname{dom} F$. We only consider (iii) for Υ, the inequality for Γ being analogous with $\nabla Q(u)$ replaced by b. We thus need to prove for all $u \in U$ that

$$\|w^k - \bar w\|^2 \geq \gamma_B \|w^{k+1} - \bar w\|^2 - C_Q \|u^{k+1} - \bar u\|^2 - \pi_B \|x^k - \bar x\|^2. \tag{4.5}$$

Using (2.13) with $A_{\bar x}\bar w = -Q'(\bar u)$ and $A_{x^k}\bar w = N_{x^k}\bar w + M_{x^k}\bar w$, we expand

$$\begin{aligned} w^{k+1} - \bar w &= N_{x^k}^{-1}(-\nabla Q(u^{k+1}) - M_{x^k} w^k) - \bar w \\ &= N_{x^k}^{-1}[\nabla Q(\bar u) - \nabla Q(u^{k+1})] + N_{x^k}^{-1}(A_{\bar x} - A_{x^k})\bar w - N_{x^k}^{-1} M_{x^k}(w^k - \bar w). \end{aligned}$$

Expanding $\|w^{k+1} - \bar w\|^2$ and applying Young's inequality thrice yields

$$\begin{aligned} \|w^{k+1} - \bar w\|^2 \leq & \left(1 + \frac{\alpha^2\gamma_B}{\lambda(1-\alpha^2\gamma_B)} + \beta\right) \|N_{x^k}^{-1}(A_{\bar x} - A_{x^k})\bar w\|^2 + \frac{1}{\alpha^2\gamma_B}\|N_{x^k}^{-1} M_{x^k}(w^k - \bar w)\|^2 \\ &+ \left(\frac{1+\beta}{\beta} + \frac{\alpha^2\gamma_B}{(1-\lambda)(1-\alpha^2\gamma_B)}\right) \|N_{x^k}^{-1}[\nabla Q(u^{k+1}) - \nabla Q(\bar u)]\|^2. \end{aligned}$$

The conditions (4.3) and (4.4) establish $\|N_{x^k}^{-1}(A_{\bar x} - A_{x^k})\bar w\|^2 \leq \gamma_N^{-2} L_A^2 \|\bar w\|^2 \|\bar x - x^k\|^2$, $\|N_{x^k}^{-1}[\nabla Q(u^{k+1}) - \nabla Q(\bar u)]\|^2 \leq \gamma_N^{-2} L_Q^2 \|u^{k+1} - \bar u\|^2$, and $\|N_{x^k}^{-1} M_{x^k}(w^k - \bar w)\|^2 \leq \alpha^2\gamma_B \|w^k - \bar w\|^2$. Taking π_B and C_Q as stated, we therefore obtain (4.5). ◻

Example 4.6 (Jacobi splitting). If N_x is the diagonal of A_x, we obtain Jacobi splitting. The first part of (4.4) reduces to strict diagonal dominance, see [9, §10.1]. The second part always holds and N_x is invertible when the diagonal of A_x has only positive entries. Then γ_N is the minimum of the diagonal values.

Example 4.7 (Gauss–Seidel splitting). If N_x is the lower triangle and diagonal of A_x, we obtain Gauss–Seidel splitting. The first part of (4.4) holds for some $\alpha \in [0, 1)$ when A_x is symmetric and positive definite; compare [9, proof of Theorem 10.1.2]. The second part holds for some γ_N when N_x is invertible.

Example 4.8 (No splitting). If $N_x = A_x$, (4.4) holds with $\alpha = 0$ and γ_N the minimal eigenvalue of A_x, assumed symmetric positive definite. For any $\gamma_B > 1$ and $\beta > 0$, we can take $\pi_B = (1 + \beta)\gamma_B\gamma_N^{-2}L_A^2\|\bar w\|^2$ and $C_Q = (1 + \beta^{-1})\gamma_B\gamma_N^{-2}L_Q^2$.

Quasi-conjugate gradients Motivated by the conjugate gradient method for solving $A_x u = b$, see, e.g., [9], we propose to perform on Line 3 of Algorithm 2.1, and analogously Line 4 the quasi-conjugate gradient update

$$\begin{cases} r^k := b - A_{x^k} u^k, \\ z^{k+1} := -\langle p^k, A_{x^k} r^k\rangle / \|p^k\|^2_{A_{x^k}}, \\ p^{k+1} := r^k + z^{k+1} p^k, \\ t^{k+1} := \langle p^{k+1}, r^k\rangle / \|p^{k+1}\|^2_{A_{x^k}}, \\ u^{k+1} := u^k + t^{k+1} p^{k+1}. \end{cases} \tag{4.6}$$

For standard conjugate gradients $A_{x^k} \equiv A$ permits a recursive residual update optimization that we are unable to perform. We have $\langle A_{x^k} p^{k+1}, p^k \rangle = 0$ for all k, although no "A-conjugacy" relationship necessarily exists between p^{k+1} and p^j for $j < k$.

The next lemma molds the updates (4.6) into our overall framework.

Lemma 4.9. *The update (4.6) corresponds to Line 3 of Algorithm 2.1 with*

$$\Gamma_k(u, \cdot, x) = \left[\mathrm{Id} - \|p^{k+1}\|_{A_x}^{-2} A_x \left(p^{k+1} \otimes p^{k+1}\right)\right] (A_x u^k - b) \quad (u \in U). \tag{4.7}$$

for $p^{k+1} = r_x^k + z_x^{k+1} p^k$ *for* $z_x^{k+1} = -\langle p^k, A_x r_x^k \rangle / \|p^k\|_{A_x}^2$ *and* $r_x^k := A_x u^k - b$.

Proof. Indeed, expanding t^{k+1}, the u-update of (4.6) may be rewritten as

$$u^{k+1} - u^k = \|p^{k+1}\|_{A_{x^k}}^{-2} (p^{k+1} \otimes p^{k+1}) r^k.$$

Applying the invertible matrix A_{x^k} and expanding r^k, this is

$$A_{x^k}(u^{k+1} - u^k) = -\|p^{k+1}\|_{A_{x^k}}^{-2} A_{x^k} (p^{k+1} \otimes p^{k+1})(A_{x^k} u^k - b),$$

and, adding $A_{x^k} u^k - b$ on both sides, further

$$A_{x^k} u^{k+1} - b = [\mathrm{Id} - \|p^{k+1}\|_{x^k}^{-2} A_{x^k} (p^{k+1} \otimes p^{k+1})](A_{x^k} u^k - b).$$

Since $B(u^{k+1}, \cdot\,; x^k) = \langle A_{x^k} u^{k+1}, \cdot \rangle$, and $L(\cdot) = \langle b, \cdot \rangle$, the claim follows. ◻

Unless A_x is independent of x, a simple approach as in Theorem 4.5 can only verify Assumption 3.4 with $\gamma_B < 1$. We hence leave the verification of convergence of Algorithm 2.1 with quasi-conjugate gradient updates to future research.

5 NUMERICAL RESULTS

We now illustrate the numerical performance of Algorithm 2.1. We first describe our experimental setup, and then discuss the results.

5.1 EXPERIMENTAL SETUP

The PDEs in our numerical experiments take one of the forms of Section 4.1 on the domain $\Omega = [0, 1] \times [0, 1]$ with nonhomogeneous Dirichlet boundary conditions. We discretize the domain as a regular grid and the PDEs by backward differences. We use both a coarse and a fine grid.

The function G and the PDE vary by experiment, but in each one we take the regularization term for the control parameter x and the data fitting term as

$$F(x) := \frac{\alpha}{2}\|x\|_{L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Omega)}^2 + \delta_{[\lambda,\lambda^{-1}]}(x) \quad \text{and} \quad Q(u) := \widehat{\beta} \sum_{i=1}^m \|u_i - z_i\|_{L^2(\Omega)}^2 \tag{5.1}$$

for some $\alpha, \beta, \lambda > 0$ as well as $\widehat{\beta} := \beta / (2\|\bar{z}\|_{L^2(\Omega)}^2)$ where $\bar{z} = \frac{1}{m}\sum_{i=1}^m z_i$ is the average of the measurement data z_i. The variables u_i correspond to multiple copies of the same PDE with different boundary conditions $u_i = f_i$ on $\partial\Omega$, $(i = 1, \ldots, m)$, for the same control x. Parametrizing $\partial\Omega$ by $\rho : (0, 1) \to \partial\Omega$, we take as boundary data

$$f_{2j-1}(\rho(t)) = \cos(2\pi j t) \quad \text{and} \quad f_{2j}(\rho(t)) = \sin(2\pi j t), \quad (j = 1, \ldots, m/2).$$

To produce the synthetic measurement z_i, we solve for $\hat{u}_i$ the PDE corresponding to the experiment with the ground truth control parameter $\hat{x} = (\hat{A}, \hat{c})$ and boundary data f_i. To this we add Gaussian noise of standard deviation $0.01\|\hat{u}_i\|_{L^2(\Omega)}$ to get z_i.

Regarding convergence, our F is α-strongly convex. We will take convex $G = 0$ or $G = \|\cdot\|_1$. Since both have full domains, Assumption 3.1(i) holds. We have $f_i \in H^{1/2}(\partial\Omega)$ which with $\delta_{[\lambda,\lambda^{-1}]}$ in F ensures the conditions of Lemma 4.1(ii′). In our discretised setting the conditions of (iii′) and (iv′) also hold, hence Lemma 4.1 verifies Assumption 3.1. Assumption 3.4 is ensured through Example 4.6, 4.7, or 4.8. The constants from by both assumptions are tied together in the condition (3.14b). It is, however, not feasible to verify this without knowledge of the solutions $\bar{u}$ and $\bar{w}$. Our fixed step lengths verify $\tau\sigma\|K\|^2 < 1$, hence (3.14a). However, we use fixed $\omega = 1$ (i.e., $\tilde{\gamma}_F = 0$), so are in the weak convergence setting of Remark 3.12.

We next describe the PDEs for each of our experiments.

Experiment 1 (Scalar coefficient). In our first numerical experiment, we aim to determine the scalar coefficient $c \in \mathbb{R}$ for the PDEs

$$\begin{cases} -\Delta u_i + cu_i = 0 & \text{in } \Omega, \\ u_i = f_i & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $i = 1, \ldots, m$. For this problem we choose $G(Kx) = 0$. Thus the objective is

$$\min_{u,c} J(x) := \frac{\alpha}{2}\|c\mathbf{1}\|^2_{L^2(\Omega)} + \delta_{[\lambda,\lambda^{-1}]}(c) + \widehat{\beta}\sum_{i=1}^{m}\|u_i - z_i\|^2_{L^2(\Omega)} \quad \text{subject to (5.2)}. \tag{5.3}$$

Our parameter choices can be found in Table 1.

With $u = (u_1, \ldots, u_m) \in H^1(\Omega)^m$, $w = (w_{1,\Omega}, \ldots, w_{m,\Omega}, w_{1,\partial}, \ldots, w_{m,\partial}) \in W := H^1_0(\Omega)^m \times H^{1/2}(\partial\Omega)^m$, for the weak formulation of (5.2) we take

$$B(u, w; c) = \sum_{i=1}^{m}\left(\langle\nabla u_i, \nabla w_{i,\Omega}\rangle_{L^2(\Omega)} + c\langle u_i, w_{i,\Omega}\rangle_{L^2(\Omega)} + \langle \operatorname{trace}_{\partial\Omega} u_i, w_{i,\partial}\rangle_{L^2(\partial\Omega)}\right)$$

and

$$Lw = \sum_{i=1}^{m}\langle f_i, w_{i,\partial}\rangle_{L^2(\partial\Omega)}. \tag{5.4}$$

Then $B_x(u, w) = \sum_{i=1}^{m}\langle u_i, w_{i,\Omega}\rangle_{L^2(\Omega)}$ following Example 4.4.

For data generation we take $\hat{c} = 1.0$. Since we are dealing with an ill-posed inverse problem, an optimal control parameter $\bar{c}$ for (5.3) does not in general equal $\hat{c}$. Therefore, to compare algorithm progress, we take as surrogate for the unknown $\bar{c}$ the iterate $\tilde{c}_A := c^{50,000}$ on the coarse grid and $\tilde{c}_B := c^{500,000}$ on the fine grid, each computed using Algorithm 2.1 without splitting.

Experiment 2 (Diffusion + scalar coefficient). In this experiment we aim to determine the coefficient function $a : \Omega \to \mathbb{R}$ and scalar $c \in \mathbb{R}$ for the group of PDEs

$$\begin{cases} -\nabla\cdot(a\nabla u_i) + cu_i = 0 & \text{in } \Omega, \\ u_i = f_i & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

where $i = 1, \ldots, m$. The optimization problem then is

$$\min_{x=(a,c)} J(x) = \delta_{[\lambda,\lambda^{-1}]}(x) + \widehat{\beta}\sum_{i=1}^{m}\|u_i - z_i\|^2_{L^2(\Omega)} + \gamma\|\nabla a\|_1 \quad \text{subject to (5.5)}.$$

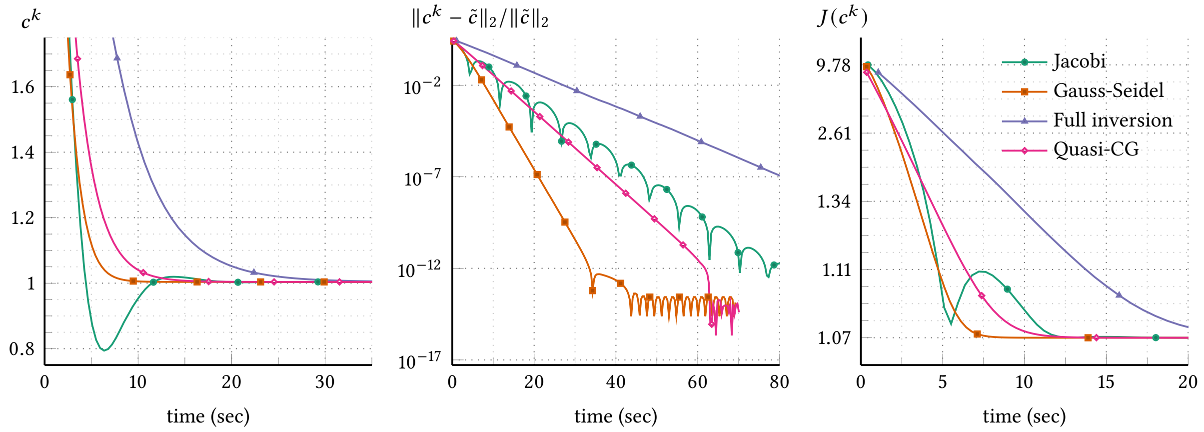

Figure 1: Performance of various splittings in the coarse grid Experiment 1.

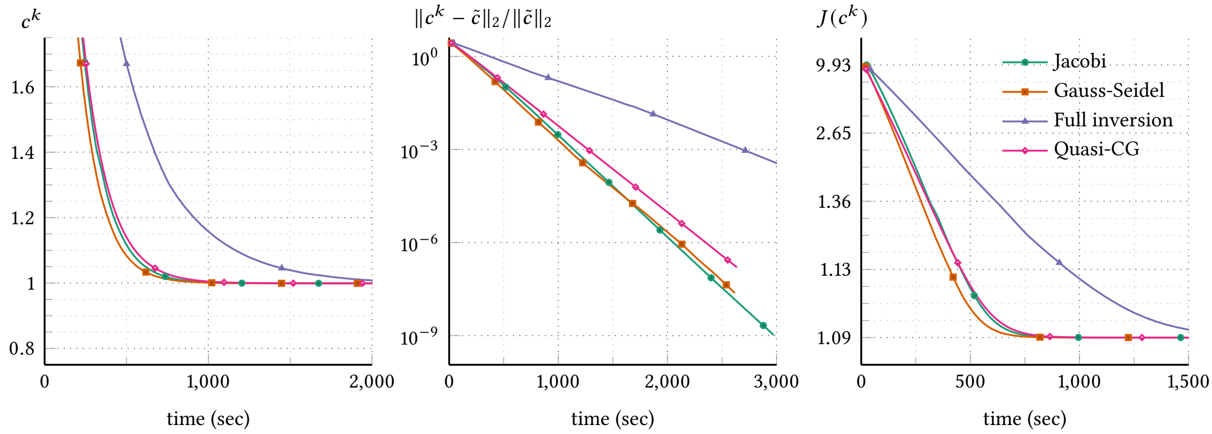

Figure 2: Performance of various splittings in fine grid Experiment 1.

Due to the total variation term, we need to ensure that a is in H^1 through discretisation. Moreover, for any control (a, c), corresponding solutions u_i to (5.5) solves the PDEs for (ta, tc) for any $t > 0$. In our progress plots we therefore consider $\frac{a}{c}$.

For the weak formulation of (5.5) with $w = (w_{1,\Omega}, \ldots, w_{m,\Omega}, w_{1,\partial}, \ldots, w_{m,\partial}) \in W := H_0^1(\Omega)^m \times H^{1/2}(\partial\Omega)^m$, $u = (u_1, \ldots, u_m) \in H^1(\Omega)^m$, and $x = (a, c) \in X \subset L^2(\Omega) \times \mathbb{R}$, we take L as in (5.4) and

$$B(u, w; x) = \sum_{i=1}^{m} \left(\langle \nabla u_i, a \nabla w_{i,\Omega} \rangle_{L^2(\Omega)} + c \langle u_i, w_{i,\Omega} \rangle_{L^2(\Omega)} + \langle \operatorname{trace}_{\partial\Omega} u_i, w_{i,\partial} \rangle_{L^2(\partial\Omega)} \right).$$

Then $B_x(u, w) = (B_x^1(w, u), B_x^2(w, u))$ takes on a mixed form with $B_x^1(w, u) = \sum_{i=1}^m \nabla u_i \cdot \nabla w_{i,\Omega}$ from Example 4.3 and $B_x^2(w, u) = \sum_{i=1}^m \langle u_i, w_{i,\Omega} \rangle_{L^2(\Omega)}$ from Example 4.4.

For data generation we take $\hat{c} = 1.0$ and $\hat{a}$ as the phantom in Figure 5. Similarly to Experiment 1 we compare the progress towards $\tilde{a} := a^{1{,}000{,}000}$ and $\tilde{c} := c^{1{,}000{,}000}$ computed using Algorithm 2.1 with full matrix inversion.

5.2 ALGORITHM PARAMETRISATION

We apply Algorithm 2.1 with no splitting (full inversion), and with Jacobi and Gauss–Seidel splitting, and quasi conjugate gradients, as discussed in Section 4.2. We fix $\sigma = 1.0$, $\omega = 1.0$, $\lambda = 0.1$, $\varepsilon = 0.01$, and

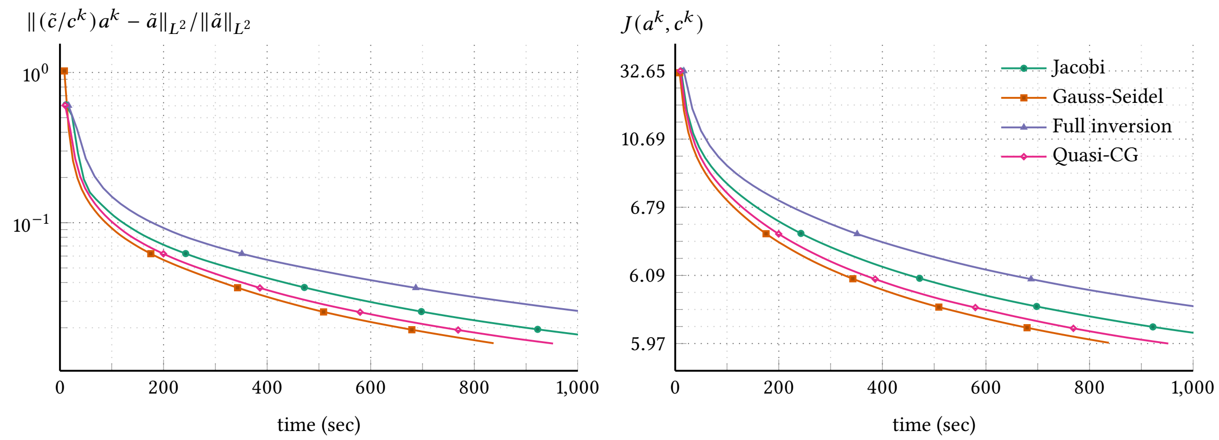

Figure 3: Performance of various splittings in the coarse grid Experiment 2.

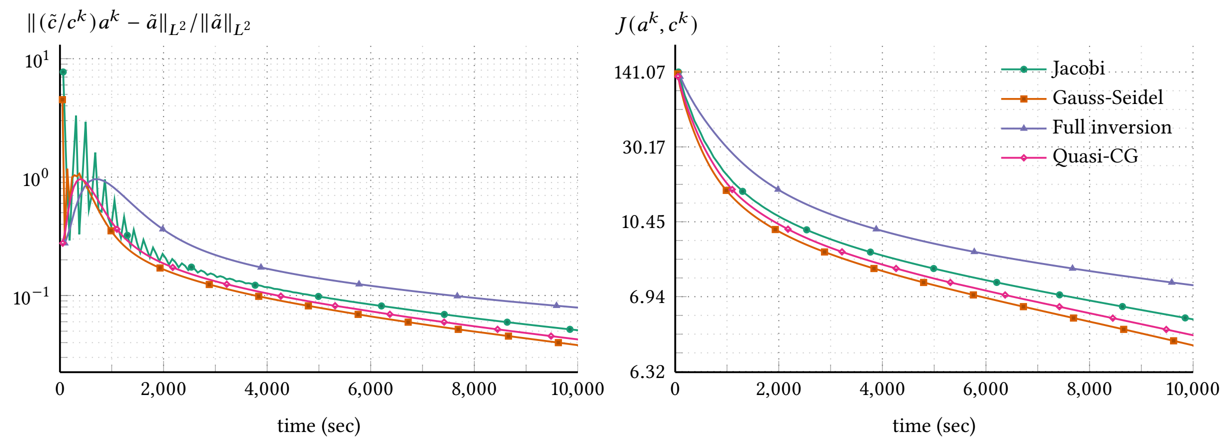

Figure 4: Performance of various splittings in the fine grid Experiment 2.

$\beta = 10^2$ for all experiments. Other parameters, including the grid size, α, γ_i, τ and m vary according to experiment with values listed in Table 1.

For the initial iterate (x^0, u^0, w^0, y^0) we make an experiment-specific choice of the control parameter x^0. Then we determine u^0 by solving the PDE, and w_0 by solving the adjoint PDE. We set $y^0 = Kx^0$. For Experiment 1 we take the initial $c^0 = 4.0$ and run the algorithm for 20,000 iterations on the coarse grid and 125,000 on the fine. For Experiment 2 we take the initial $a^0 \equiv 1.0$ a constant function, and $c^0 = 2.0$. The algorithm is run for 200,000 iterations on the coarse grid, and 500,000 on the fine.

We implemented the algorithm in Julia. The experiments were run on a ThinkPad laptop with Intel Core i5-8265U CPU at 1.60GHz ×4 and 15.3 GiB memory.

Table 1: Parameter choices for all examples.

Grid	N	Grid size	α	β	γ	τ	σ	ω	m
Coarse	51	2601	1×10^{-5}	1×10^2	0	2.5×10^{-2}	1	1	6
Fine	101	10201	1×10^{-5}	1×10^2	0	2.0×10^{-3}	1	1	6
Coarse	51	2601	0	1×10^2	10^{-2}	2.5×10^{-2}	1	1	10
Fine	101	10201	0	1×10^2	10^{-2}	1×10^{-2}	1	1	10

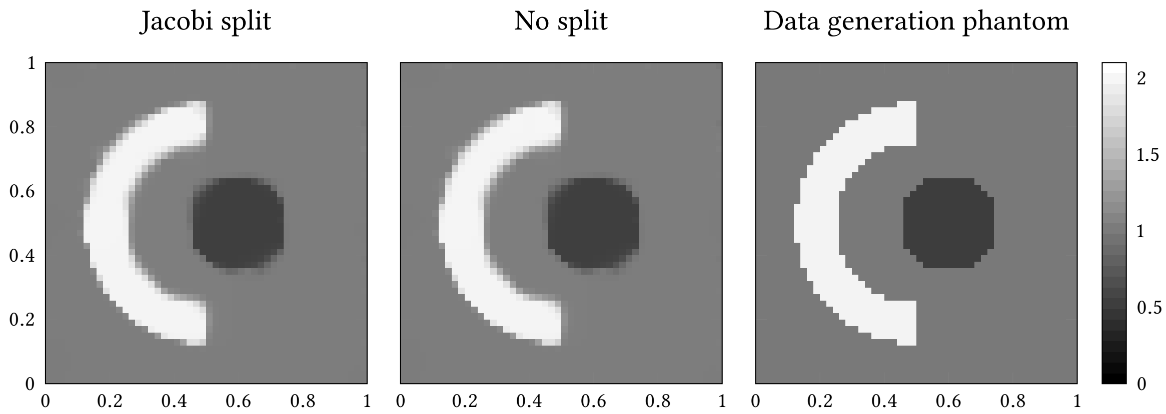

Figure 5: Illustrations of the coefficient reconstructions for Experiment 2A. On the left is the result of the Jacobi split approach, in the middle the full matrix inversion after the same number of iterations. On the right we show the data generation phantom for comparison.

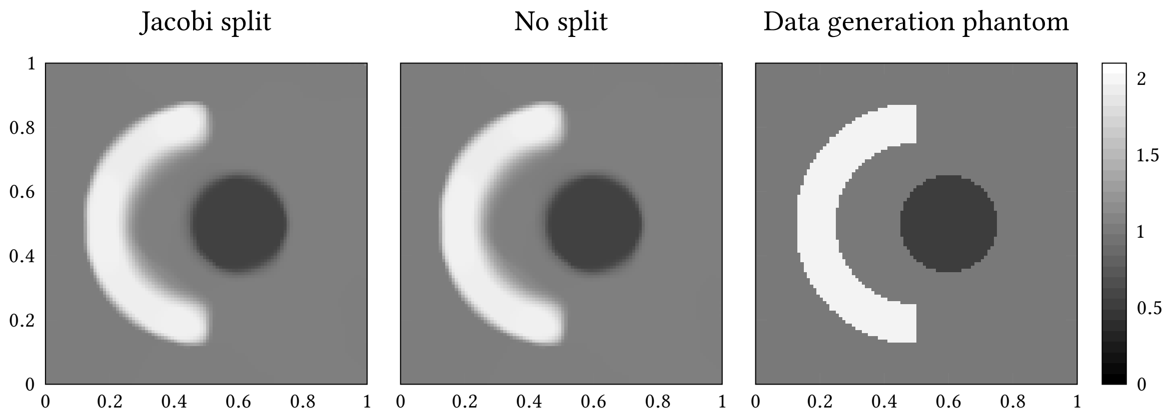

Figure 6: Illustrations of the coefficient reconstructions for Experiment 2B. On the left is the result of the Jacobi split approach, in the middle the full matrix inversion after the same number of iterations. On the right we show the data generation phantom for comparison.

5.3 RESULTS

The results for Experiment 1 with the above algorithm parametrisations are in Figure 1 for the coarse grid and Figure 2 for the fin grid. In the figures we illustrate the evolution of the coefficient c^k as the algorithm iterates. We also show the evolution of the relative error of the coefficient and the functional value.

The results for Experiment 2 are available in Figures 3 and 5 for the coarse grid and Figures 4 and 6 for the find grid. In Figures 3 and 4 are shown the evolution of the relative error of the coefficient and the functional value. In Figures 5 and 6 are the reconstructed coefficients a^k at the final iterates and for comparison the phantom used for the data generation.

The performance plots have *time* on the x-axis rather than the number of iterations, as the main difference between the splittings is expected to be in the computational effort for linear system solution, i.e., Lines 3 and 4 of Algorithm 2.1. For fairness, we limited the number of threads used by Julia/OpenBLAS to one.

In all experiments the splittings outperform full matrix inversion: the best splittings require roughly *half of the computational effort* for an iterate of the same quality. No particular splitting completely dominates another, however, Jacobi appear to be more prone to overstepping and oscillatory patterns. On the other hand, quasi-CG currently has no convergence theory, and we have observed situations where it does not exhibit convergence while Jacobi and Gauss–Seidel splittings do. Therefore, Gauss–Seidel is our recommended option.

APPENDIX A LOCALIZATION

Theorems 3.10 and 3.11 are global convergence results, but also depend on the global constant C_x in Assumption 3.1 (iv). To satisfy the conditions of the theorems, $\operatorname{dom} F$ may need to be small for C_x to be small. We now develop local convergence results that allow replacing C_x by a small initialization-dependent value without restricting $\operatorname{dom} F$.

We replace Assumption 3.1 with the following:

Assumption A.1. We assume Assumption 3.1 to hold with (iv) replaced by

(iv′) For some $\tilde{C}_x \geq 0$, for all $(u, w) \in U \times W$ and $x \in \operatorname{dom} F$ we have the bound

$$B_{\text{lin}}(u, w; x - \bar{x}) \leq \tilde{C}_x \|x - \bar{x}\|_X \|u\|_U \|w\|_W.$$

This estimate uses the standard norm in X, which is a 2-norm in the examples of Sections 4.1 and 5. However, Section 4.1 gives estimates involving an ∞-norm for C_x. Therefore some finite-dimensionality of the parameters is required to satisfy Assumption A.1 (iv′). This can take the form of a finite element discretisation of a function parameter a, or the parameter being a scalar constant. In the latter case, the examples of Section 4.1 readily verify Assumption A.1.

We then modify several previous results accordingly:

Lemma A.2 (Local version of Lemma 3.6). *Let $k \in \mathbb{N}$. Suppose Assumptions 3.3, 3.4 and A.1 hold,*

$$\|u^{k+1} - \bar{u}\|_U \leq \delta_{uw}, \quad \textit{and} \quad \|w^{k+1} - \bar{w}\|_U \leq \delta_{uw}, \tag{A.1}$$

for some $\delta_{uw} > 0$, *and for some* $\varepsilon_u, \varepsilon_w, \mu > 0$ *that*

$$\gamma_F \geq \tilde\gamma_F + \varepsilon_u + \varepsilon_w + \frac{\lambda_{k+1} + \theta_{k+1}}{\eta_k}\pi_B, \tag{A.2a}$$

$$\gamma_{G^*} \geq \tilde\gamma_{G^*}, \tag{A.2b}$$

$$\gamma_B \geq \frac{\lambda_{k+1}}{\lambda_k} + \frac{\theta_k}{\lambda_k}C_Q + \frac{2\eta_k \mathcal{S}(\bar w)}{\varepsilon_w \lambda_k} + \frac{\tilde C_x^2 \delta_{uw}^2 \mu \eta_k}{4\varepsilon_u \lambda_k}, \quad \text{and} \tag{A.2c}$$

$$\gamma_B \geq \frac{\theta_{k+1}}{\theta_k} + \frac{2\eta_k \mathcal{S}(\bar u)}{\varepsilon_u \theta_k} + \frac{\tilde C_x^2 \delta_{uw}^2 \eta_k}{4\varepsilon_w \mu \theta_k}. \tag{A.2d}$$

Then (3.6) *holds.*

Proof. We follow the proof of Lemma 3.6 until the estimate (3.8), which now holds with $C_x = \tilde C_x\|x-\bar x\|_X$ and any $\tilde\varepsilon_u, \tilde\varepsilon_w, \tilde\mu > 0$ standing for $\varepsilon_u, \varepsilon_w, \mu > 0$. Recall that we abbreviate $u = u^{k+1}$, $w = w^{k+1}$, and $x = x^{k+1}$. Using Young's inequality and (A.1), we continue from there estimating that

$$\begin{aligned}
&\eta_k\langle B_x(u,w) - B_x(\bar u, \bar w), x - \bar x\rangle \\
&\geq -\eta_k\left(\frac{\mathcal{S}(\bar u)}{\tilde\varepsilon_u} + \frac{\tilde C_x\|x-\bar x\|\tilde\mu}{2}\right)\|w-\bar w\|_W^2 - \eta_k\left(\frac{\mathcal{S}(\bar w)}{\tilde\varepsilon_w} + \frac{\tilde C_x\|x-\bar x\|}{2\tilde\mu}\right)\|u-\bar u\|_U^2 \\
&\quad - \eta_k(\tilde\varepsilon_u + \tilde\varepsilon_w)\|x-\bar x\|_X^2 \\
&\geq -\eta_k\left(\frac{\mathcal{S}(\bar u)}{\tilde\varepsilon_u} + \frac{\tilde C_x^2\delta_{uw}^2\tilde\mu^2}{8\tilde\varepsilon_u}\right)\|w-\bar w\|_W^2 - \eta_k\left(\frac{\mathcal{S}(\bar w)}{\tilde\varepsilon_w} + \frac{\tilde C_x^2\delta_{uw}^2}{8\tilde\mu^2\tilde\varepsilon_w}\right)\|u-\bar u\|_U^2 \\
&\quad - \eta_k(2\tilde\varepsilon_u + 2\tilde\varepsilon_w)\|x-\bar x\|_X^2.
\end{aligned}$$

With $\varepsilon_u = 2\tilde\varepsilon_u$, $\varepsilon_u = 2\tilde\varepsilon_w$, and $\mu = \tilde\mu^2$, we now continue with the proof of Lemma 3.9, which goes through with (A.2) in place of (3.5). ◻

Lemma A.3 (Local version of Lemma 3.7). *Let* $k \in \mathbb{N}$. *Suppose* $\gamma_F > \tilde\gamma_F > 0$ *as well as* $\gamma_{G^*} \geq \tilde\gamma_{G^*} \geq 0$ *and that there exists* $c > 0$ *with* $\eta_{k+1} \leq c\eta_k$ *such that*

$$\gamma_B \geq c + \frac{\mathcal{S}(\bar u)}{\mathcal{S}(\bar w)}C_Q + \frac{2c\pi_B\left(\mathcal{S}(\bar u) + \mathcal{S}(\bar w)\right)}{(\gamma_F - \tilde\gamma_F)^2}\left(8 + \frac{\tilde C_x^2\delta_{uw}^2}{2\sqrt{\mathcal{S}(\bar w)\mathcal{S}(\bar u)}}\right). \tag{A.3}$$

Then there exist $\varepsilon_u, \varepsilon_w, \mu > 0$ *such that* (A.2) *holds for the choices* λ_k *and* θ_k *given by* (3.10).

Proof. Since the proof of Lemma 3.7 takes $\varepsilon_u = \varepsilon_w = (\gamma_F - \tilde\gamma_F)/4$, replacing C_x therein by

$$\frac{\tilde C_x^2\delta_{uw}^2}{4\varepsilon_u} = \frac{\tilde C_x^2\delta_{uw}^2}{4\varepsilon_w} = \frac{\tilde C_x^2\delta_{uw}^2}{\gamma_F - \tilde\gamma_F},$$

it goes through with (A.2) in place of (3.5) and (A.3) in place of (3.9). Observe that compared to (3.5c) and (3.5d), (A.2c) and (A.2d) have an additional factor 2 in front of the terms involving ε_u and ε_w. This difference produces a similar difference in the constant $8 = 2 \cdot 4$ in (A.3) compared to (3.9). ◻

Lemma A.4 (Local version of Lemma 3.9). *Suppose Assumptions 3.3 and A.1 hold as do Assumption 3.4 and* (A.2) *for* $k = 0, \ldots, N-1$ *with*

$$\delta_{uw}^2 = \frac{1}{\gamma_B}\max\left\{\frac{1}{\lambda_0}, \frac{C_Q\gamma_B^{-1}}{\lambda_0}, \frac{1}{\theta_0}, \frac{(1 + C_Q\gamma_B^{-1})\pi_B}{\varphi_0(1-\kappa) + (\lambda_0 + \theta_0)\pi_B}\right\}\delta^2 \tag{A.4}$$

and

$$\delta := \|v^0 - \bar v\|_{Z_0 \tilde M_0}. \tag{A.5}$$

Also suppose $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{\theta_k\}_{k\in\mathbb{N}}$ are non-decreasing. Given v^0, let $v^1, \ldots, v^{N-1}$ be produced by Algorithm 2.1. Then (3.12) holds for $k = 0, \ldots, N-1$, where all the terms are non-negative.

Proof. We need to prove (A.1) for all $k = 0, \ldots, N-1$. The rest follows as in the proof of Lemma 3.9.

Assumption 3.4 (iii) with (3.11) and Lemma 3.5 establish for all $k = 0, \ldots, N-1$ the *a priori* bounds

$$\begin{aligned}
\|u^{k+1} - \bar u\|_U^2 &\le \frac{1}{\gamma_B}\left(\|u^k - \bar u\|_U^2 + \pi_B \|x^k - \bar x\|_X^2\right) \\
&\le \frac{1}{\gamma_B} \max\left\{\frac{1}{\lambda_k}, \frac{\pi_B}{\varphi_k(1-\kappa) + (\lambda_k + \theta_k)\pi_B}\right\} \|v^k - \bar v\|_{Z_k \tilde M_k}^2 \\
&\le \frac{1}{\gamma_B} \max\left\{\frac{1}{\lambda_0}, \frac{\pi_B}{\varphi_0(1-\kappa) + (\lambda_0 + \theta_0)\pi_B}\right\} \|v^k - \bar v\|_{Z_k \tilde M_k}^2 \\
&\le \frac{\delta_{uw}^2}{\delta^2} \|v^k - \bar v\|_{Z_k \tilde M_k}^2
\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}
\|w^{k+1} - \bar w\|_W^2 &\le \frac{1}{\gamma_B}\left(\|w^k - \bar w\|_W^2 + C_Q \|u^{k+1} - \bar u\|_U^2 + \pi_B \|x^k - \bar x\|_X^2\right) \\
&\le \frac{1}{\gamma_B}\left(\|w^k - \bar w\|_W^2 + C_Q\gamma_B^{-1} \|u^k - \bar u\|_U^2 + (1 + C_Q\gamma_B^{-1})\pi_B \|x^k - \bar x\|_X^2\right) \\
&\le \frac{1}{\gamma_B} \max\left\{\frac{1}{\theta_k}, \frac{C_Q\gamma_B^{-1}}{\lambda_k}, \frac{(1 + C_Q\gamma_B^{-1})\pi_B}{\varphi_k(1-\kappa) + (\lambda_k + \theta_k)\pi_B}\right\} \|v^k - \bar v\|_{Z_k \tilde M_k}^2 \\
&\le \frac{1}{\gamma_B} \max\left\{\frac{1}{\theta_0}, \frac{C_Q\gamma_B^{-1}}{\lambda_0}, \frac{(1 + C_Q\gamma_B^{-1})\pi_B}{\varphi_0(1-\kappa) + (\lambda_0 + \theta_0)\pi_B}\right\} \|v^k - \bar v\|_{Z_k \tilde M_k}^2 \\
&\le \frac{\delta_{uw}^2}{\delta^2} \|v^k - \bar v\|_{Z_k \tilde M_k}^2 .
\end{aligned} \tag{A.7}$$

In the final steps we have used the the assumptions that $\{\varphi_k\}_{k\in\mathbb{N}}$ (by Assumption 3.3), $\{\lambda_k\}_{k\in\mathbb{N}}$, and $\{\theta_k\}_{k\in\mathbb{N}}$ are non-decreasing.

We now use induction. By definition we have $\|v^0 - \bar v\|_{Z_0 \tilde M_0} \le \delta$. Hence (A.6) and (A.7) verify (A.1) for $k = 0$. Suppose then that we have proved (A.1) for $k = 0, \ldots, \ell - 1$. Then (3.12) holds $k = 0, \ldots, \ell - 1$ by following the proof of Lemma 3.9, replacing Lemma 3.6 there in by the localized Lemma A.2. Summing (3.12) over $k = 0, \ldots, \ell - 1$, we now obtain the *a posteriori* bound

$$\frac{1}{2}\|v^\ell - \bar v\|_{Z_\ell \tilde M_\ell}^2 \le \frac{1}{2}\|v^0 - \bar v\|_{Z_0 \tilde M_0}^2 = \frac{1}{2}\delta^2.$$

Now (A.6) and (A.7) verify (A.1) for $k = \ell$. Hence also (3.12) holds for $k = \ell$. As a result of the entire inductive argument, it holds for all $k = 0 \ldots, N-1$. □

With $\varphi_0 = 1$ and the choices of $\lambda_0 = a\mathcal{S}(\bar w)\tau_0$ and $\theta_0 = a\mathcal{S}(\bar u)\tau_0$ in (3.10), where

$$a := \frac{\gamma_F - \tilde\gamma_F}{2c\pi_B\left(\mathcal{S}(\bar u) + \mathcal{S}(\bar w)\right)},$$

we expand (A.4) as

$$\delta_{uw}^2 = \frac{1}{\gamma_B} \max\left\{\frac{\max\{1, C_Q\gamma_B^{-1}\}}{a\tau_0\mathcal{S}(\bar w)}, \frac{1}{a\tau_0\mathcal{S}(\bar u)}, \frac{(1 + C_Q\gamma_B^{-1})\pi_B}{(1-\kappa) + a\tau_0(\mathcal{S}(\bar w) + \mathcal{S}(\bar u))\pi_B}\right\} \delta^2. \tag{A.8}$$

We can estimate

$$\begin{aligned}\delta_{uw}^2 &\le \frac{1}{a\tau_0\gamma_B}\max\left\{\frac{\max\{1, C_Q\gamma_B^{-1}\}}{\mathcal{S}(\bar w)}, \frac{1}{\mathcal{S}(\bar u)}, \frac{1+C_Q\gamma_B^{-1}}{\mathcal{S}(\bar w)+\mathcal{S}(\bar u)}\right\}\delta^2\\ &\le \frac{2c\pi_B\left(\mathcal{S}(\bar u)+\mathcal{S}(\bar w)\right)}{\tau_0\gamma_B(\gamma_F-\tilde\gamma_F)}\max\left\{\frac{\max\{1, C_Q\gamma_B^{-1}\}}{\mathcal{S}(\bar w)}, \frac{1}{\mathcal{S}(\bar u)}, \frac{1+C_Q\gamma_B^{-1}}{\mathcal{S}(\bar w)+\mathcal{S}(\bar u)}\right\}\delta^2.\end{aligned} \tag{A.9}$$

Hence (A.3) with δ_{uw}^2 replaced by this upper estimate reads

$$\gamma_B \ge c + \frac{\mathcal{S}(\bar u)}{\mathcal{S}(\bar w)}C_Q + \frac{16c\pi_B\left(\mathcal{S}(\bar u)+\mathcal{S}(\bar w)\right)}{(\gamma_F-\tilde\gamma_F)^2} + \frac{\tilde C_x^2\max\left\{\frac{\max\{1,C_Q\gamma_B^{-1}\}}{\mathcal{S}(\bar w)}, \frac{1}{\mathcal{S}(\bar u)}, \frac{1+C_Q\gamma_B^{-1}}{\mathcal{S}(\bar w)+\mathcal{S}(\bar u)}\right\}}{2(\gamma_F-\tilde\gamma_F)\sqrt{\mathcal{S}(\bar w)\mathcal{S}(\bar u)}}\delta^2, \tag{A.10a}$$

where we recall that

$$\delta := \|v^0 - \bar v\|_{Z_0\tilde M_0}. \tag{A.10b}$$

We now immediately obtain local versions of the main results. By initializing close enough to a solution, i.e., with small δ, we can possibly obtain convergence more often than from the global versions.

Corollary A.5 (Local accelerated convergence). *In Theorem 3.10, replace Assumption 3.1 by Assumption A.1 and (3.14b) by (A.10) with $c = \omega_0^{-1}$. Then the claims continue to hold.*

Corollary A.6 (Local linear convergence). *In Theorem 3.11, replace Assumption 3.1 by Assumption A.1 and (3.17b) and (A.10) with $c = \omega^{-1}$. Then the claims continue to hold.*

Both proofs are exactly as the original proofs, using Lemma A.4 in place of Lemma 3.9.

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