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Extension of primal-dual interior point methods to diff-convex problems on symmetric cones

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We consider the extension of primal dual interior point methods for linear programming on symmetric cones, to a wider class of problems that includes approximate necessary optimality conditions for functions expressible as the difference of two convex functions of a special form. Our analysis applies the Jordan-algebraic approach to symmetric cones. As the basic method is local, we apply the idea of the filter method for a globalisation strategy.

Keywords: Symmetric cone, Jordan algebra, diff-convex, filter method, interior point

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1. Introduction

Consider the convex function

$$f(y) := \sup\{\langle B^*y + c, p\rangle \mid p \in \mathcal{K}, Ap = b\}, \tag{1}$$

where $\mathcal{K}$ is (the closure of) a symmetric cone, and A is a linear mapping, such that the constraint set for p is non-empty and bounded. The necessary and sufficient optimality conditions for this class of functions may be expressed as

$$B^*y + A^*\lambda + d + c = 0,\ \ Ap = b, Bp = 0,\ \ p \circ d = 0,\ \ p, d \in \mathcal{K}, \tag{2}$$

with $\circ$ denoting a Jordan algebra product. This set of equations belongs to the same class as those for linear programs on symmetric cones, and very efficient algorithms exist for approximately solving such equations; cf. e.g. [1, 8, 9, 18, 20, 25, 26] in more general cases, and [3, 23, 32] in the special case of Euclidean norms, various sums of which are included in the class (1).

We study the extension of these methods to so-called ϵ-semi-criticality conditions for functions expressible as the difference of functions of class (1). Examples of such problems include the multi-source Weber problem or K-spatial-medians, another clustering objective from [30], as well the reformulations of the Euclidean travelling salesperson problem from [29]. Also included are the Weber problem with attraction and repulsion [4], along with the Euclidean Steiner tree problem.

The aforementioned extension faces the problem that the linearised perturbed version of (2) may become singular, something that does not occur in the convex

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case under rather mild assumptions. We briefly present an analysis of such situations (more details may be found in [28]), through which we obtain an alternative derivation of the perturbed version of conditions (2) – often also derived through the use of barrier functions. Additionally we obtain an alternative interpretation of what an "interior point" is, which, it could be said, makes our approach "graphical programming".

As our extension is, however, not globally convergent due to the above-mentioned singularities, we also study globalisation strategies. One approach is that of a filter method, following the line of research initiated in [12]. These methods crucially depend on so-called restoration methods that restore feasibility after the main filter method – which will presently be a variation of the interior point method – runs into trouble. We therefore derive and analyse one based on the simple (and common) idea of sequential convex programming (SCP). Although we provide complete descriptions of these algorithms, and prove their convergence, at this stage of development, they do not yet offer competitive practical performance, and further development more practically-oriented is needed. This paper rather offers an initial survey of some possibilities of extending the Jordan algebraic interior point method approach to DC problems, and provides a range of theoretical results relevant to further work on the algorithms.

Aside from the general literature on interior point methods (see e.g. [13, 22] and references therein), and the already-cited papers on linear programs over symmetric cones, the work in [10] and [33] bears some relationship to ours, in extending the Jordan-algebraic approach. In the former, quadratic programs with symmetric cone constraints are considered using a potential reduction approach, while in the latter general non-linear programs with second-order cone constraints are considered employing merit functions under C^2 assumptions. Within filter methods, the works most related to ours appear to be in particular [27], as well as [31] in having an interior point approach. However, the research on filter methods so far, has concentrated on constrained programming, whereas we apply the idea to two optimality criteria related to the original unconstrained objective function.

The rest of this paper is organised as follows. Section 2 introduces the basic notation used in this paper, and contains a quick introduction to the Jordan-algebraic machinery used. After that, in Section 3, we study the objective function in some detail; however due to space constraints, a lot of the proofs are omitted and the reader referred to the author's Ph.D thesis [28]. In Section 4 the primal-dual interior point method is developed and its convergence rate analysed. In Section 5 we discuss another method that merely applies interior point methods to convex sub-problems. This method is then applied in Section 6 as a restoration method of the discussed globalisation strategy. Finally, some preliminary practical experience is presented and the paper concluded in Section 7.

2. Preliminaries

2.1. *Sets and mappings*

First we introduce some basic notation. Let A be a mapping. Then $\mathcal{R}(A)$ denotes its range. When A is also linear, $\mathcal{N}(A)$ denotes its null-space. The adjoint of a linear operator A between two inner product spaces is denoted by A^*, and the pseudoinverse by $A^\dagger$. For two mappings, $(A, B)(x, y) := (Ax, By)$, and $(A; B)(x, y) := Ax + By$.

Let then C be a set. Its interior is denoted by $\operatorname{int} C$, the relative interior by $\operatorname{ri} C$, and the border by $\operatorname{bd} C$.

Following [24], recall that the (contingent) *tangent cone* to a set $C \subset \mathbb{R}^m$ at

$x \in C$ is defined as

$$T_C(x) := \limsup_{\tau \searrow 0} (C - x)/\tau = \{\Delta x \mid x + \tau \Delta x' \in C, \tau \searrow 0, \Delta x' \to \Delta x\}.$$

This agrees with the tangent cone of convex analysis in the convex case, justifying the notation. Taking the tangent to the graph of a set-valued function S at (y, z), $z \in S(y)$, we get the (contingent) graphical derivative

$$\begin{aligned} DS(y|z)(\Delta y) &:= \{\Delta z \mid (\Delta y, \Delta z) \in T_{\mathrm{Graph}\, S}(y, z)\} \\ &= \{\Delta z \mid z + \tau \Delta z' \in S(y + \tau \Delta y'), \tau \searrow 0, (\Delta y', \Delta z') \to (\Delta y, \Delta z)\}. \end{aligned}$$

2.2. *Euclidean Jordan algebras*

In this subsection we introduce the bare minimum of the theory of (finite-dimensional Euclidean) Jordan algebras necessary for the analysis of this paper. We will rely on the Jordan algebra of quadratic forms related to the familiar second-order cone as a concrete example in our exposition. More detailed treatment may be found in e.g. [7, 17].

A (real) Jordan algebra $\mathcal{J}$ is a real vector space endowed with a multiplication operator $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, that is bilinear, commutative, and satisfies the property

$$x \circ (x^2 \circ y) = x^2 \circ (x \circ y) \text{ where } x^2 = x \circ x.$$

We assume in addition that $\mathcal{J}$ is *Euclidean* (or *formally real*), satisfying: $x^2 + y^2 = 0$ implies $x = y = 0$.

Then $\mathcal{J}$ has a multiplicative unit element e ($x \circ e = x$). An element x is called invertible, if there exists an element x^{-1}, such that $x \circ x^{-1} = x^{-1} \circ x = e$. We denote by $L(x)$ the symmetric linear operator $(x \circ \cdot) : \mathcal{J} \to \mathcal{J}$. $L(x)$ is invertible precisely when x is. We say that x and y *operator-commute* when $L(x)L(y) = L(y)L(x)$.

An element c is called an *idempotent*, if $c \circ c = c$. It is *primitive*, if it cannot be composed by summing from other idempotents. A *complete orthogonal system of primitive idempotents* or a *Jordan frame* $c_1, \ldots, c_r$ is such that $c_i \circ c_j = 0$ for $i \neq j$, and $\sum_{j=1}^r c_j = e$. The number r is the *rank* of $\mathcal{J}$.

It turns out that for each $x \in \mathcal{J}$, there exist unique real numbers $\zeta_1, \ldots, \zeta_r$, called the *eigenvalues* of x, and a Jordan frame $c_1, \ldots, c_r$, such that $x = \sum_{j=1}^r \zeta_i c_i$. If all the eigenvalues are positive, x is called *positive-definite*. The number of non-zero eigenvalues is the *rank* of x. Powers of x may be defined as $x^\alpha := \sum_j \zeta_i^\alpha c_i$ when meaningful. We may also define the determinant $\det x := \prod_j \zeta_j$, and the trace $\operatorname{tr} x := \sum_j \zeta_j$.

The trace may be used to define the inner product $\langle x, y \rangle := \operatorname{tr}(x \circ y)$, which is positive-definite and associative, satisfying $\langle L(x)y, z \rangle = \langle y, L(x)z \rangle$. We may also define the norms $\|x\|_F := \sqrt{\sum_j \zeta_j^2} = \sqrt{\langle x, x \rangle}$ and $\|x\|_2 := \max_j |\zeta_j|$. According to [25, Lemma 4], we have $\|x \circ y\|_F \le \|x\|_2 \|y\|_F \le \|x\|_F \|y\|_F$.

The *quadratic presentation* of x is defined as $Q_x := 2L(x)^2 - L(x^2)$. It turns out that the invertibility of x is equivalent to the invertibility of Q_x as well. Important properties, which can be found in [26], include $Q_x^k = Q_{x^k}$, $Q_{Q_x y} = Q_x Q_y Q_x$, $Q_x x^{-1} = x$, and $Q_x e = x^2$.

Also denote $Q_{x,y} := L(x)L(y) + L(y)L(x) - L(x \circ y)$. Then $Q_x = Q_{x,x}$. For a Jordan frame $c_1, \ldots, c_r$, $Q_{c_i, c_j} = 2L(c_i)L(c_j) = 2L(c_j)L(c_i)$ for $i \neq j$, and the operators Q_{c_i} ($i = 1 \ldots r$) and $2Q_{c_i, c_j}$ ($i < j$) form a complete set of orthogonal projection operators in $\mathcal{J}$. More precisely, $\mathcal{R}(Q_{c_i}) = \{x \mid L(c_i)x = x\} = \mathbb{R}c_i$ and $\mathcal{R}(Q_{c_i,c_j}) = \{x \mid L(c_i)x = L(c_j)x = x/2\}$ for $i \neq j$, as follows from the theory of *Peirce decompositions.* If $x = \sum_{i=1}^{r} \zeta_i c_i$, then $L(x) = \sum_i \zeta_i Q_{c_i} + \sum_{i<j}(\zeta_i + \zeta_j)Q_{c_i,c_j} = \sum_{i,j}(\zeta_i + \zeta_j)Q_{c_i,c_j}/2$.

Example 2.1 Consider the space $\mathcal{E}_{m+1}$ of $m+1$ element vectors $x = (x^0, \bar{x})$ with $x^0 \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^m$. Define the operator $\circ$ on $\mathcal{E}_{m+1}$ as

$$x \circ y = (x^T y, x^0 \bar{y} + y^0 \bar{x}).$$

Then $(\mathcal{E}_{m+1}, \circ)$ is a Euclidean Jordan algebra with inner product $\langle x, y\rangle = 2x^T y$, identity $e = (1, 0)$, and rank $r = 2$. The operator $L(x)$ is given by

$$L(x) = \operatorname{Arw}(x) := \begin{bmatrix} x^0 & \bar{x}^T \\ \bar{x} & x^0 I \end{bmatrix}$$

with I the identity matrix. Denote $R := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -I \end{smallmatrix}\right]$. Then $\det x = x^T R x = (x^0)^2 - \|\bar{x}\|^2$, and $x^{-1} = Rx/\det x$ when $\det x \neq 0$.

2.3. *Symmetric cones*

The *cone of squares* of $\mathcal{J}$ is defined as $\mathcal{K} = \mathcal{K}(\mathcal{J}) := \{x^2 \mid x \in \mathcal{J}\}$. It turns out that the cones generated this way are precisely the so-called symmetric cones, and are the same as the self-scaled cones of Nesterov and Todd [20]. Important properties include [7, 17]

(i) $\operatorname{int}\mathcal{K} = \{x \in \mathcal{J} \mid x \text{ is positive-definite}\} = \{x \in \mathcal{J} \mid L(x) \text{ pos. def.}\}$.
(ii) $\langle x, y\rangle \geq 0$ for all $y \in \mathcal{K}$ iff $x \in \mathcal{K}$, and
(iii) $\langle x, y\rangle > 0$ for all $y \in \mathcal{K} \setminus \{0\}$ iff $x \in \operatorname{int}\mathcal{K}$.
(iv) Q_x for $x \in \operatorname{int}\mathcal{K}$ maps $\mathcal{K}$ onto itself.
(v) For $x, y \in \operatorname{int}\mathcal{K}$, there is a unique $a \in \operatorname{int}\mathcal{K}$, such that $x = Q_a y$.
(vi) For any $x, y \in \mathcal{K}$, $\langle x, y\rangle = 0$ iff $x \circ y = 0$ [9].

In relation to (barrier) interior point methods, the following properties are particularly important:

(vii) $B(x) := -\log(\det x)$ tends to infinity as x goes to $\operatorname{bd}\mathcal{K}$.
(viii) $\nabla B(x) = -x^{-1}$, $\nabla^2 B(x) = Q_x$ when differentiated wrt. $\langle\cdot,\cdot\rangle$.
(ix) $\|y\|_x := \|Q_x^{-1/2} y\|_F$ defines a local norm around $x \in \operatorname{int}\mathcal{K}$, such that $\|y - x\|_x = \|Q_x^{-1/2} y - e\|_F \leq 1$ implies $y \in \mathcal{K}$. (This follows by considering the eigenvalue definition of $\|\cdot\|_F$, and the onto-property of Q_x; cf. also [20].)

Example 2.2 For the Jordan algebra of quadratic forms $\mathcal{E}_{m+1}$, we get the so-called second order cone, $\mathcal{K} = \{x \mid x^0 \geq \|\bar{x}\|\}$.

Definition 2.3: We say that two elements $p, d \in \mathcal{K}$ are *strictly complementary,* if $p \circ d = 0$, and $p + d \in \operatorname{int}\mathcal{K}$ [21, 26].

Lemma 2.4: *Suppose that p, d are strictly complementary. Then $p \circ \Delta d + d \circ \Delta p = 0$ iff $(\Delta p, \Delta d) = (L(p)\eta, -L(d)\eta)$ for some $\eta \in \mathcal{J}$.*

Proof: Since $p \circ d = 0$, there exists a common Jordan frame $c_1, \ldots, c_r$ and eigenvalues $\zeta_1, \ldots, \zeta_r \geq 0$ and $\sigma_1, \ldots, \sigma_r \geq 0$ with $\zeta_i \sigma_i = 0$ and $\zeta_i + \sigma_i > 0$, such that $p = \sum_i \zeta_i c_i$, and $d = \sum_i \sigma_i c_i$. Therefore, recalling the representation of

$L(p) = \sum_{i,j}(\zeta_i + \zeta_j)Q_{c_i,c_j}/2$ and $L(d) = \sum_{i,j}(\sigma_i + \sigma_j)Q_{c_i,c_j}/2$, we have

$$L(p)\Delta d + L(d)\Delta p = 0 \iff Q_{c_i,c_j}\big((\zeta_i + \zeta_j)\Delta d + (\sigma_i + \sigma_j)\Delta p\big) = 0 \text{ for all } i, j.$$

Note that always either $\zeta_i + \zeta_j > 0$ or $\sigma_i + \sigma_j > 0$, so that $\zeta_i + \zeta_j = 0$ forces $Q_{c_i,c_j}\Delta p = 0$, and the other way around. Consequently, Δp is proportional to Δd on $\mathcal{R}(Q_{c_i,c_j})$. Therefore $\Delta p, \Delta d \propto Q_{c_i,c_j}\eta$ for some $\eta \in \mathcal{J}$, which may be chosen the same for all i, j by orthogonality of the projection operators Q_{c_i,c_j}. The correct proportionality factors are given by the choice $\Delta p = L(p)\eta$ and $\Delta d = -L(d)\eta$ for some $\eta \in \mathcal{J}$.

On the other hand, strictly complementary p and d operator-commute (as seen from the Q-decomposition of L; cf. [26, Theorem 27]), so the equality follows from the representation. □

3. The objective function

3.1. *A class of convex functions*

We will now consider convex functions on $\mathbb{R}^m$ of the form (1). That is,

$$f(y) := \sup\{\langle B^*y + c, p\rangle \mid p \in \mathcal{K}, Ap = b\} = \sigma_V(B^*y + c), \tag{3}$$

where $\mathcal{K}$ is a symmetric cone with associated Jordan algebra $\mathcal{J}$, $A : \mathcal{J} \to \mathbb{R}^{m_A}$ and $B : \mathcal{J} \to \mathbb{R}^m$ ($m > 0$) are linear mappings, $c \in \mathcal{J}$, $V := \{p \in \mathcal{K} \mid Ap = b\}$, and σ_V is the support function of V. We require that

$$\mathcal{N}(B^*; A^*) = \{0\}, \tag{4}$$

$$\mathcal{N}(A) \cap \mathcal{K} = \{0\}, \text{ and} \tag{5}$$

$$b \in A(\operatorname{int}\mathcal{K}). \tag{6}$$

Example 3.1

(i) If $\mathcal{K}$ is the second-order cone on $\mathcal{E}_{m+1}$, $Ap := p^0 = \langle e/2, p\rangle$ (recalling that the inner product on $\mathcal{E}_{m+1}$ is two times the standard $\mathbb{R}^{m+1}$ inner product) $b := 1$, $c := (0, -a/2)$, and $Bp := \bar{p}$ (whence $B^*y = (0, y/2)$), we get $f(y) = \sup\{(y - a)^T\bar{p} \mid 1 = p^0 \geq \|\bar{p}\|\} = \|y - a\|$.

(ii) Weighted sums $\sum_{k=1}^n \|W_k(y - a_k)\|$ of Euclidean norms can be represented by a straightforward extension: $p = (p_1, \ldots, p_n) \in \mathcal{K}^n$, $Ap := (p_1^0, \ldots, p_n^0)$, $b \equiv 1$, $B^*y := ((0, W_1y), \ldots, (0, W_ny))/2$, and $c := -((0, W_1a_1), \ldots, (0, W_na_n))/2$.

(iii) Finally, if we instead set $Ap := \sum_{k=1}^n p_k^0$ and $b = 1$, the supremum favours maximum $\langle W_k(y - a_k), \bar{p}\rangle$. We therefore have $f(y) = \max_{k=1,\ldots,n} \|W_k(y - a_k)\|$. Likewise, $\max_{j=1,\ldots,n} \sum_{k\neq j} \|W_k(y-a_k)\|$ may be presented in the form (3) with the help of slack variables ($A(p, \theta) := (p_1^0+\theta_1, \ldots, p_n^0+\theta_n, \sum_i \theta_i) \equiv 1$, $\theta_i \in [0, \infty) = \mathcal{E}_{0+1}$, etc.).

Lemma 3.2: *Assumptions* (4)–(6) *imply*

(i) V is bounded with non-empty relative interior.

(ii) If $A^*\lambda \in \mathcal{K}$ and $\langle b, \lambda\rangle = 0$, then $\lambda = 0$.

Proof: (i) There is a $p_b \in \operatorname{int}\mathcal{K}$ such that $Ap_b = b$. Also $\mathcal{N}(A) \neq \{0\}$ by (4). Thus $V = (p_b + \mathcal{N}(A)) \cap \mathcal{K}$ has non-empty relative interior. A standard argument employing (5) shows boundedness.

(ii) Let U be a neighbourhood of p_b in $\mathcal{K}$. We have $\langle p, A^*\lambda\rangle \geq 0$ for all $p \in \mathcal{K}$, and in particular $\langle p_b, A^*\lambda\rangle = \langle b, \lambda\rangle = 0$. But then, since $\mathcal{N}(A^*) = \{0\}$, there is a $p' \in U$ with $\langle p', A^*\lambda\rangle < 0$ unless $\lambda = 0$. □

Recall that the ϵ-subdifferential of a convex function f with respect to the inner product $\langle\cdot,\cdot\rangle$ is defined as [15]

$$\partial_\epsilon f(y) := \{z \mid f(y') - f(y) \geq \langle z, y' - y\rangle - \epsilon \text{ for all } y'\}.$$

Since the relative interior of V is non-empty by Lemma 3.2 above, the tools of convex analysis (see, e.g., [15, Chapter XI]) yield for f defined by (3) that

$$\begin{aligned}\partial_\epsilon f(y) &= B\partial_\epsilon \sigma_V(B^*y + c)\\ &= \{Bp \mid \langle p, d\rangle \leq \epsilon, Ap = b, B^*y + A^*\lambda + d + c = 0,\ p, d \in \mathcal{K}\}.\end{aligned} \tag{7}$$

Remark 3.1 : The set of equations for $0 \in \partial_\epsilon f(y)$ are very similar to the standard primal-dual equations for barrier methods, but without an explicit central path ($p \circ d = \mu e$) selected. Indeed, let $f^\mu(y) := \sup_{p\in V}\{\langle B^*y + c, p\rangle + \mu \log(\det p)\}$ be a barrier-smoothing of f. It is differentiable because $\log(\det p)$ is strictly concave in $\operatorname{int}\mathcal{K}$ (with $\nabla^2 \log(\det p) = -Q_p$), and we have $\nabla f^\mu(y) = B\{p \in V \mid B^*y + c + \mu p^{-1} \in -N_V(p)\} = \{Bp \mid Ap = b, B^*y + A^*\lambda + c + d = 0, p \circ d = \mu e, p, d \in \mathcal{K}\}$, using $d = \mu p^{-1}$.

After we look at the difference of functions of type (3) shortly, we will be doing some second-order analysis, where we need the following notion of non-degeneracy. Conditions ensuring this will be further discussed in Section 3.4.

Definition 3.3: We say that a strictly complementary pair (p, d) is *non-degenerate relative to a subspace* $X \subset \mathcal{J}$, if $(L(d)\eta, L(p)\eta) \in \mathcal{R}(A^*) \times (X \cap \mathcal{N}(A))$ implies $\eta = 0$.

Example 3.4 Consider the base case of Example 3.1. At $y = a$, we have $d = 0$ and strict complementarity holds for $p = (1, \bar{p})$ with $\|\bar{p}\| < 1$. As $L(p)$ is non-singular, (p, d) is not non-degenerate (relative to $\mathcal{J}$), but it is non-degenerate relative to $\mathcal{N}(B) = \mathbb{R}e = \mathcal{R}(A^*)$.

3.2. *Taking the difference*

Let f be of the class (3), and subscript the data and variables as B_f, A_f, c_f, b_f, $\mathcal{K}_f$, etc. Let ν be another function in this class, with similar subscripts. Now let $f_\nu := f - \nu$, making f_ν a diff-convex function; see e.g. [14, 16] for an overview of the theory of this general class of functions.

Example 3.5 Recalling from Example 3.1 that sums and maxima of (matrix-scaled) Euclidean distances can be represented in the form (3), we find that e.g. the multisource Weber problem objective function

$$(y_1, \ldots, y_K) \mapsto \sum_{k=1}^{n} \min_i \|a_k - y_i\| = \sum_{k=1}^{n}\sum_{i=1}^{K} \|a_k - y_i\| - \sum_{k=1}^{n} \max_{j=1,\ldots,K} \sum_{i\neq j} \|a_k - y_i\|$$

has the form f_ν. So does the clustering objective $\bar{y} \mapsto \sum_{k=1}^n \sum_{i=1}^K \|a_k - y_i\| - \lambda\sum_{i<j}\|y_i - y_j\|$ considered in [30], as well as the reformulations of the Euclidean TSP based on the above clustering criteria in [29].

Our objective is then to minimise f_ν, or at least find an approximately critical point. Recall that when ν is strictly differentiable, $\partial^\circ f_\nu(y) = \partial f(y) - \nabla\nu(y)$ for the subdifferential of Clarke [5], among others. But at other points this does not necessarily hold. Nevertheless, in trying to minimise f_ν, we may have to content ourselves with *semi-critical* points (cf. [2, 30]), where it holds

$$0 \in \partial f(y) - \partial\nu(y) \quad \text{or, equivalently} \quad \partial f(y) \cap \partial\nu(y) \neq \emptyset.$$

It is natural to extend this definition to ϵ-*semi-critical* points, mimicking the ϵ-subdifferential formula for sums of convex functions (cf. [15])

$$0 \in \partial_\epsilon^{\mathrm{DC}} f_\nu(y) := \bigcup \{\partial_{\epsilon_1} f(y) - \partial_{\epsilon_2}\nu(y) \mid \epsilon_1 + \epsilon_2 = \epsilon, \epsilon_1, \epsilon_2 \geq 0\}.$$

Now, note that the condition

$$\operatorname{tr} p_f \circ d_f \leq \epsilon_1 \text{ and } \operatorname{tr} p_\nu \circ d_\nu \leq \epsilon_2 \text{ for some } \epsilon_1 + \epsilon_2 = \epsilon, \epsilon_1, \epsilon_2 \geq 0$$

reduces to $\operatorname{tr}(p_f, p_\nu) \circ (d_f, d_\nu) \leq \epsilon$ in the product algebra $\mathcal{J} := \mathcal{J}_f \times \mathcal{J}_\nu$. Thus, recalling the representation of $\partial_\epsilon f$ from (7), we actually get with $A := (A_f, A_\nu)$, $B := (B_f; B_\nu)$, $B_- := (B_f; -B_\nu)$, $c := (c_f, c_\nu)$, and $b := (b_f, b_\nu)$ that

$$\partial_\epsilon^{\mathrm{DC}} f_\nu(y) = \{B_- p \mid Ap = b,\ B^* y + A^*\lambda + d + c = 0,\ \operatorname{tr} p \circ d \leq \epsilon,\ p, d \in \mathcal{K}\}.$$

Note that the non-degeneracy condition relative to $\mathcal{N}(B)$ is equivalent to that relative to $\mathcal{N}(B_-)$: supposing it did not hold for one, replacing $\eta = (\eta_f, \eta_\nu)$ with $(\eta_f, -\eta_\nu)$ in the definition, shows that it does not hold for the other, for $L(p)\eta \in \mathcal{N}(A)$ and $L(d)\eta \in \mathcal{R}(A^*)$ are unaffected by such change.

3.3. *Second order behaviour*

In order to derive a second-order or Newton-type method for minimising f_ν, we now study the second order derivative. Because of possible non-differentiability of f_ν, we employ "graphical" notions of differentiation from [24], which we have briefly introduced in Section 2.

Let begin by setting

$$\begin{aligned} S_\epsilon &:= \{(p, d) \in \mathcal{K} \times \mathcal{K} \mid Ap = b,\ B^* y + A^*\lambda + c + d = 0,\ \operatorname{tr} p \circ d \leq \epsilon\}, \\ G(p, d) &:= ((I; 0)(B^*; A^*)^\dagger(-d - c), B_- p), \quad \text{and} \\ G_\epsilon^{-1}(v) &:= \{(p, d) \in S_\epsilon \mid G(p, d) = v\}. \end{aligned} \tag{8}$$

Then $\partial_\epsilon^{\mathrm{DC}} f_\nu(y) = \{z \mid (y, z) \in GS_\epsilon\}$, and it is possible to show (see [28])

$$D(\partial_\epsilon^{\mathrm{DC}} f_\nu)(y|z)(\Delta y) \supset \bigcup_{(p,d) \in G_\epsilon^{-1}(y,z)} \{\Delta z \mid (\Delta y, \Delta z) \in \nabla G(p, d) T_{S_\epsilon}(p, d)\}$$

with equality for $\epsilon = 0$ when all $(p, d) \in G_\epsilon^{-1}(y, z)$ are strictly complementary and non-degenerate relative to $\mathcal{N}(B_-)$.

The information in $D(\partial_\epsilon^{\mathrm{DC}} f_\nu)$ is not quite sufficient for our needs, however, so we extend it. More specifically, we let $\widehat{G}(p,d) := (G(p,d), p \circ d)$,

$$\widehat{G}_\epsilon^{-1}(y,z,q) := \{(p,d) \in S_\epsilon \mid \widehat{G}(p,d) = (y,z,q)\},$$

and consider

$$\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu(y) := \{(z,q) \mid (y,z,q) \in \widehat{G}S_\epsilon\}.$$

We may regard the q-component of $\widehat{\partial}^{\mathrm{DC}} f_\nu$ as indicating a specific "selection" $y \mapsto \{z \mid (z,q) \in \widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu(y)\}$ within $\partial_\epsilon^{\mathrm{DC}} f_\nu$, approximating the differences of subgradients of f and ν. In particular, the selections $q = (\epsilon/r)e$ give the gradients of barrier-approximations to f_ν; see Remark 3.1. So $\partial_\epsilon^{\mathrm{DC}} f_\nu$ is then a bundle with the information of the particular approximation lost, whereas $\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu$ retains that information. $D(\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu)(y|z,q)$ then combines the gradient of a selection with inter-selection differential information.

The following assumption will be used frequently in what follows. Conditions ensuring the stated requirements will be further discussed in Section 3.4. Note that it may happen that $p \circ d \notin \mathcal{K}$.

Assumption 3.6: Let $(z,q) \in \widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu(y)$. Then $q = 0$ (resp. $q \in \mathcal{K} \setminus \{0\}$) and all $(p,d) \in \widehat{G}_\epsilon^{-1}(y,z,q)$ are strictly complementary and non-degenerate relative to $\mathcal{N}(B_-)$ (resp. $p, d \in \operatorname{int} \mathcal{K}$).

We now get the following results. Due to space constraints, we refer to [28] for the proofs.

Theorem 3.7: *Suppose Assumption 3.6 holds and $\epsilon > 0$ (resp. $\epsilon = 0$). Then*

$$(\Delta z, \Delta q) \in D(\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu)(y|z,q)(\Delta y)$$

if and only if for some $(p,d) \in \widehat{G}_\epsilon^{-1}(y,z,q)$ and $(\Delta p, \Delta d, \Delta\lambda)$, we have

$$\operatorname{tr} \Delta q \leq \infty(\epsilon - \operatorname{tr} p \circ d) \quad (\textit{resp. } \Delta q = 0), \tag{9}$$

$$B^* \Delta y + A^* \Delta\lambda + \Delta d = 0, \tag{10}$$

$$A \Delta p = 0, \tag{11}$$

$$B_- \Delta p = \Delta z, \tag{12}$$

$$p \circ \Delta d + d \circ \Delta p = \Delta q. \tag{13}$$

Moreover, for $\epsilon = 0$, we have $\widehat{\partial}^{\mathrm{DC}} f_\nu(y) = \partial^{\mathrm{DC}} f_\nu(y) \times \{0\}$.

In the interior point methods that we will develop in the next section, it is of importance to know when we can solve $(0, \Delta q) \in D(\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu)(y|z,q)(\Delta y)$ for Δy with fixed Δq, along with obtaining $(\Delta p, \Delta d)$. The following result provides one condition towards that end.

Lemma 3.8: *Suppose Assumption 3.6 holds along with the following second order condition: $0 \in D(\widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu)(y|z,q)(\Delta y)$ implies $\Delta y = 0$. Then the system (10)–(13) is solvable for $(\Delta p, \Delta d, \Delta y, \Delta\lambda)$ in a neighbourhood (in $\mathcal{K} \times \mathcal{K}$) of $(p,d) \in \widehat{G}_\epsilon^{-1}(y,z,q)$.*

Remark 3.2: When f_ν is twice continuously differentiable at y, we have $D(\partial^{\mathrm{DC}} f_\nu)(y|z)(\Delta y) = \nabla^2 f_\nu(y) \Delta y$. Thus the second order condition reduces to non-singularity of the Hessian.

3.4. *Non-degeneracy*

The following results ensure relative non-degeneracy, uniqueness, and Assumption 3.6. We often use

Assumption 3.9: $\mathcal{K} = \prod_{i=1}^{m_A} \mathcal{K}_i$ for symmetric cones $\mathcal{K}_i$ (in a Jordan algebra $\mathcal{J}_i$ of rank r_i), and $Ap = (\langle a_1', p_1\rangle, \ldots, \langle a_{m_A}', p_{m_A}\rangle)$ with $a_i' \in \operatorname{int} \mathcal{K}_i$ when $p = (p_1, \ldots, p_{m_A})$, $p_i \in \mathcal{K}_i$.

Lemma 3.10: *Suppose Assumption 3.9 holds and* $b > 0$. *Then* (5)–(6) *hold, and* $(p, d) \in S_0$ *and* $L(d)\eta \in \mathcal{R}(A^*)$ *imply* $L(d)\eta = 0$.

Proof: Assumptions (5)–(6) are immediate from the form of A. If $L(d)\eta = A^*\lambda$, we may assume $\lambda \geq 0$: by the independence of $L(d)$ on the sub-algebras corresponding to the $\mathcal{K}_i$, by negating components, we could find such a $\lambda' \geq 0$ and η' for which this holds. Therefore, unless $\lambda = 0$,

$$\langle a_i', p_i\rangle = b_i > 0 \tag{14}$$

implies $0 < \langle b, \lambda\rangle = \langle p, A^*\lambda\rangle = \langle p, L(d)\eta\rangle = \langle p \circ d, \eta\rangle = 0$. This is a contradiction, whence $L(d)\eta = 0$. □

Lemma 3.11: *Suppose that* $B_- p = (\sum_i W_{1i} p_i, \ldots, \sum_i W_{Ni} p_i)$ *in addition to Assumption 3.9. Let* B_-' *denote* B_- *with those* W_{ji} *removed,* $(j = 1, \ldots, N)$ *for which* d_i *has rank* $r_i - 1$, $(i = 1, \ldots, m_A)$. *Denote by* A' *the corresponding modification of* A. *Then* $L(p)\eta = 0$ *if*

$$\mathcal{N}(A') \cap \mathcal{N}(B_-') = \{0\}, \tag{15}$$

$(p, d) \in S_0$, $L(d)\eta = 0$, *and* $L(p)\eta \in \mathcal{N}(A) \cap \mathcal{N}(B_-)$. *Consequently, strict complementarity of* $(p, d) \in S_0$ *and* (15) *imply non-degeneracy wrt.* $\mathcal{N}(B_-)$.

Proof: If d_i has rank $r_i - 1$, then p_i is proportional to a single primitive idempotent c complementary to d_i. This and $L(d_i)\eta_i = 0$ imply that $\eta_i \in \mathcal{R}(Q_c^*) = \mathcal{R}(Q_c) = \mathbb{R}c$ (as can be seen from the Q-decomposition of $L(p)$). Consequently $s_i := L(p_i)\eta_i \propto p_i$. But then $s_i \in \pm\mathcal{K}_i$, which is in contradiction to $\langle a_i', s_i\rangle = 0$ unless $s_i = 0$, since $a_i' \in \operatorname{int} \mathcal{K}_i$. Therefore $L(p_i)\eta_i = 0$, and we may consequently remove the corresponding terms from the equations $B_- L(p)\eta = 0$ and $AL(p)\eta = 0$. The resulting equation has no non-zero solution when $\mathcal{N}(A') \cap \mathcal{N}(B_-') = \{0\}$.

As for the final claim, Lemma 3.10 reduces the non-degeneracy requirement relative to $\mathcal{N}(B_-)$ into $(L(d)\eta, L(p)\eta) \in \{0\} \times (\mathcal{N}(A) \cap \mathcal{N}(B_-))$ implying $\eta = 0$. Since $L(d + p)$ is invertible when p and d are strictly complementary, it suffices to show that $L(p)\eta = 0$. The first part of this lemma did that. □

Corollary 3.12: *Suppose each* $\mathcal{J}_i$ *has rank* $r_i = 2$ *(i.e.* $\mathcal{K}_i$ *is isomorphic to the second order cone), and* $\mathcal{N}(W_{ji}) \cap \mathcal{N}(\langle a_i', \cdot\rangle) = \{0\}$. *Then strictly complementary* (p, d) *are non-degenerate relative to* $\mathcal{N}(B_-)$ *when for each* $j = 1, \ldots, N$, *at most one* $d_i = 0$ *with* $W_{ji} \neq 0$.

Proof: When $d_i \neq 0$, $p_i \neq 0$ is proportional to a single primitive idempotent. Consequently B_-' has just one non-zero W_{ji} on each row. By assumption $\mathcal{N}(W_{ji}) \cap \mathcal{N}(\langle a_i', \cdot\rangle) = \{0\}$, so (15) holds. Lemma 3.11 now shows non-degeneracy. □

The following results prove and simplify Assumption 3.6 through uniqueness.

Lemma 3.13: *Suppose* $(p, d) \in \widehat{G}_0^{-1}(y, z, 0)$ *is strictly complementary and non-degenerate wrt.* $\mathcal{N}(B_-)$. *Then it is unique. In particular, Assumption 3.6 holds.*

Proof: Suppose $(p+\Delta p, d+\Delta d) \in \widehat{G}_0^{-1}(y,z,0)$. Then $\Delta d \in \mathcal{R}(A^*)$ and $\Delta p \in \mathcal{N}(A) \cap \mathcal{N}(B_-)$. Consequently $\operatorname{tr} \Delta p \circ \Delta d = 0$. As $p \circ d = (p+\Delta p)\circ(d+\Delta d) = 0$, taking the trace we then find that $\operatorname{tr}(p \circ \Delta d + d \circ \Delta p) = 0$. This says that $\operatorname{tr}(p + \alpha\Delta p)\circ(d+\alpha\Delta d) = 0$ for all $\alpha \in [0,1]$. Because $p+\alpha\Delta p, d+\alpha\Delta d \in \mathcal{K}$ by convexity, we find that $(p+\alpha\Delta p)\circ(d+\alpha\Delta d) = 0$. Differentiating $(p+\alpha\Delta p)\circ(d+\alpha\Delta d)$ at $\alpha = 0$, we find $p \circ \Delta d + d \circ \Delta p = 0$. Now strict complementarity and Lemma 2.4 imply $(\Delta p, \Delta d) = (L(p)\eta, -L(d)\eta)$ for some $\eta \in \mathcal{J}$. By non-degeneracy $\eta = 0$. Therefore, (p,d) is unique. □

Lemma 3.14: *Suppose $p, d \in \mathcal{K}$ and $q = p \circ d \in \operatorname{int} \mathcal{K}$. Then $p, d \in \operatorname{int} \mathcal{K}$, so Assumption 3.6 holds.*

Proof: If $d \in \operatorname{bd} \mathcal{K}$, there is a $v \in \mathcal{K} \setminus \{0\}$ such that $v \circ d = 0$ (as this is equivalent to $\langle v, d\rangle = 0$). Now, $\langle v, q\rangle = \langle v, p \circ d\rangle = \langle v \circ d, p\rangle = 0$, in contradiction to $q \in \operatorname{int} \mathcal{K}$. The case $p \in \operatorname{bd} \mathcal{K}$ is analogous. □

Example 3.15 Suppose $f(y) = \sum_{i=1}^n \|y - c_i\|$ and $c_i \neq c_j$ for $i \neq j$. Strict complementarity implies non-degeneracy and Assumption 3.6, because at most one term is non-differentiable at a single point, with corresponding $p_i \in \operatorname{int} \mathcal{K}_i$, and $\langle a_i', p_i\rangle = p_i^0$, $W_{ji} p_i = \bar{p}_i$. Thus the linear independence condition $\mathcal{N}(W_{ji}) \cap \mathcal{N}(\langle a_i', \cdot\rangle) = \{0\}$. holds. Similar results hold for more complex combinations of norms; cf. also [23, Section 3].

3.5. *Scaling*

The following scaling invariance of the presentation of f, and by extension f_ν, holds with respect to the automorphisms of the cone $\mathcal{K}$.

Lemma 3.16: *Let f have the representation* (3)*, and let $v \in \operatorname{int} \mathcal{K}$. Define*

$$\tilde{f}(y) := \sup\{\langle \underset{\sim}{B}^* y + \underset{\sim}{c}, \tilde{p}\rangle \mid \tilde{p} \in \mathcal{K}, \underset{\sim}{A}\tilde{p} = b\}$$

with $\underset{\sim}{B} := BQ_v^{-1}$, $\underset{\sim}{A} := AQ_v^{-1}$, and $\underset{\sim}{c} = Q_v^{-1}c$. Then $\tilde{f} = f$ with $\tilde{p} = Q_v p$ producing the same value. In the representation of $\partial_\epsilon f$, same result is produced with $\underset{\sim}{d} = Q_v^{-1} d$.

Proof: Firstly note that assumptions (4)–(5) continue to hold after scaling, so $\tilde{f}$ has the required form (3). Now the claims follow in a straightforward manner from Q_v being a bijection in $\mathcal{K}$. □

Remark 3.3: The scaling invariance extends to f_ν and $\partial_\epsilon^{\mathrm{DC}} f_\nu$ in the obvious way. Note, however, that the q of $(z,q) \in \widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu(y)$ generally depends on the scaling. In the special case of the "central selection" $q = \mu e$, it is unaffected, as seen from [26, Lemma 28] for $\mu > 0$ and from the basic properties of symmetric cones for $\mu = 0$.

4. A primal-dual interior point method

4.1. *On interior point methods for the convex case*

Suppose we're given a point $0 \in \partial_\epsilon f(y)$. To minimise f, we want to reduce ϵ, while at the same time keeping the constraint $0 \in \partial_\epsilon f(y)$. Thus we want to choose a direction Δy such that $0 \in D(\partial_\epsilon f)(y|0)(\Delta y)$ and ϵ can be reduced afterwards. If $(y, 0) \in \operatorname{int} \operatorname{Graph} \partial_\epsilon f(y)$, any direction satisfies this. When we additionally want to be moving towards a "central selection" from a selection q with $(0,q) \in \widehat{\partial}_\epsilon f(y)$, we require that $(0, \Delta q) \in D(\widehat{\partial}_\epsilon f)(y|0,q)(\Delta y)$ for $\Delta q := \sigma\mu e - q$, $\mu = \mu(q) := \operatorname{tr} q/r$, and a chosen $\sigma \in (0,1)$. We may think of Δq consisting of a "tangential step" $(\sigma-1)\mu e$ aiming to reduce μ or ϵ, and a "normal step" $\mu e - q$ aiming to move closer to the central selection for $\partial_{r\mu} f$.

Suppose furthermore that we have $(p,d) \in \widehat{G}_\epsilon^{-1}(y,0,q)$, and want to make our movement in the neighbourhood of (p,d). Then $q = p \circ d$ and we arrive from $(\Delta y, 0, \sigma\mu e - q) \in \nabla\widehat{G}(p,d)T_{S_\epsilon(p,d)}$ into the system

$$
\begin{gathered}
A\Delta p = 0, B\Delta p = 0,\\
B^*\Delta y + A^*\Delta\lambda + \Delta d = 0,\\
p \circ \Delta d + d \circ \Delta p = \sigma\mu e - p \circ d,\\
\Delta p \in T_{\mathcal{K}}(p), \Delta d \in T_{\mathcal{K}}(d).
\end{gathered}
$$

When $p, d \in \operatorname{int}\mathcal{K}$ and $p \circ d \in \mathcal{K}$, the linear system is solvable. By iterating steps in directions found this way after suitable scaling and step length selection, we get the usual primal-dual interior point method for linear programs on symmetric cones; see [1, 8, 9, 18, 20, 25, 26].

Whereas typically the "interior" refers to the interior of a constraint set, and the above system of equations have been derived through either the use of barrier functions, or by perturbation of the KKT conditions, here the conditions have been derived through subdifferential analysis, and we can alternatively consider to be moving in the interior of $\partial_\epsilon f$ and even the set $\widehat{G}S_\infty = \operatorname{Graph}\widehat{\partial}_\infty f$, while maintaining the ϵ-optimality constraint $0 \in \partial_\epsilon f(y)$, reducing ϵ by a constant factor at each iteration. Additionally, we try to stay close to a "central selection" $p \circ d = \mu e$, corresponding to the differential of a smoothing of f by a barrier function.

4.2. *Solvability in the diff-convex case*

Our objective is now analogous to the convex case: given $(0,q) \in \widehat{\partial}_\epsilon^{\mathrm{DC}} f_\nu(y)$ and $(p,d) \in \widehat{G}_\epsilon^{-1}(y,0,q)$, we try to solve $(0,\Delta q) \in D(\widehat{\partial}_\epsilon^{\mathrm{DC}} f)(y|0,q)(\Delta y)$ near (p,d). When $p, d \in \operatorname{int}\mathcal{K}$ and $\Delta q = \sigma\mu e - p \circ d$, the resulting set of equations may then according to Theorem 3.7 be written

$$A\Delta p = 0, B_-\Delta p = 0, \tag{16}$$

$$B^*\Delta y + A^*\Delta\lambda + \Delta d = 0, \tag{17}$$

$$p \circ \Delta d + d \circ \Delta p = \sigma\mu e - p \circ d. \tag{18}$$

This differs from the convex case by the use of B_- instead of B in the condition for Δp. Consequently, we run into the following two problems in a direct generalisation of the methods for convex problems: (a) we may have $\langle \Delta p, \Delta d\rangle \neq 0$, and (b) the system may not have a solution for any specific value of Δq. Therefore other strategies are needed for global convergence. But let us first analyse how far a direct generalisation goes, and its convergence properties.

According to the results of Section 3.3 and Lemma 3.8 in particular, the system (16)–(18) can be solved at least locally in the neighbourhood of a point y arising from relatively non-degenerate and strictly complementary (p,d), and where $0 \in D(\widehat{\partial}^{\mathrm{DC}} f_\nu)(y|0)(\Delta y)$ implies $\Delta y = 0$. Furthermore, this second order condition reduces to non-singularity of the Hessian when f_ν is twice continuously differentiable.

Likewise, by the same lemma, the system (16)–(18) is solvable near nicelybehaving selections of $\widehat{\partial}_{\epsilon}^{\mathrm{DC}} f_\nu$. Also, since central selections $q = \mu e \in \mathcal{K}$, $\mu > 0$, are unaffected by scaling as remarked in Section 3.5, the same applies to scaled representation of f near central selections. Some further technical discussion on solvability may be found in [28].

4.3. *Neighbourhoods*

Let $P_e^\perp q := q - \langle e, q \rangle e/r$ be the projection of q to the subspace orthogonal to e. If the spectrum of q is $\{\zeta_i(q)\}$, then by the e-sum property of Jordan frames, the spectrum of $P_e^\perp q$ is $\{\zeta_i(q) - \mu(q)\}$ with $\mu(q) := \sum_j \zeta_j(q)/r = \operatorname{tr} q/r$. Now, define the distance functions

$$d_\bullet(p, d) := \|P_e^\perp Q_p^{1/2} d\|_\bullet \quad \text{and} \quad d_\bullet^*(p, d) := \|P_e^\perp (p \circ d)\|_\bullet,$$

with $\bullet \in \{F, 2, -\infty\}$ and, abusing norm notation for the sake of convenience, $\|s\|_{-\infty} := -\min_i \zeta_i(s)$. For $P_e^\perp q$ we then get $\|P_e^\perp q\|_{-\infty} = \mu(q) - \min \zeta_i(q)$, $\|P_e^\perp q\|_F = \sqrt{\sum_i (\zeta_i(q) - \mu(q))^2}$, and $\|P_e^\perp q\|_2 = \max_i |\zeta_i(q) - \mu(q)|$.

When $p, d \in \operatorname{int} \mathcal{K}$, we know from the effects of $P_e^\perp$ on the spectrum and [26, Proposition 21 and Lemma 30], that $d_\bullet(d, p) = d_\bullet(p, d) \leq d_\bullet^*(p, d) = d_\bullet^*(d, p)$ for $p, d \in \operatorname{int} \mathcal{K}$. When p and d operator-commute, equality holds as then $p \circ d = Q_p^{1/2} d$.

Now, let $\gamma \in (0, 1)$, and for $\bullet \in \{F, 2, -\infty\}$ define the corresponding short, semi-long, and long-step neighbourhoods of $\mathcal{K} \times \mathcal{K}$ as

$$\mathcal{C}_\bullet(\gamma) := \{(p, d) \in \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K} \mid d_\bullet(p, d) \leq \gamma \mu(p \circ d)\} \quad \text{and}$$
$$\mathcal{C}_\bullet^*(\gamma) := \{(p, d) \in \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K} \mid d_\bullet^*(p, d) \leq \gamma \mu(p \circ d)\},$$

We then have $\mathcal{C}_\bullet^*(\gamma) \subset \mathcal{C}_\bullet(\gamma)$, as well as $\mathcal{C}_F(\gamma) \subset \mathcal{C}_2(\gamma) \subset \mathcal{C}_{-\infty}(\gamma)$, and likewise for the starred neighbourhoods. The unstarred neighbourhoods are scaling-invariant, i.e., $(p, d) \in \mathcal{C}_\bullet(\gamma)$ implies $(\widetilde{p}, \underset{\sim}{d}) = (Q_v p, Q_v^{-1} d) \in \mathcal{C}_\bullet(\gamma)$ for $v \in \operatorname{int} \mathcal{K}$ by [26, Proposition 29]. Furthermore, a scaling that results in operator-commutative $(\widetilde{p}, \underset{\sim}{d})$ ensures that $(\widetilde{p}, \underset{\sim}{d}) \in \mathcal{C}_\bullet^*(\gamma)$ for $(p, d) \in \mathcal{C}_\bullet(\gamma)$.

In the method we keep (p, d) in an appropriate γ-neighbourhood to ensure desirable properties, such as $p \circ d \in \operatorname{int} \mathcal{K}$ (cf. Lemma 3.14).

4.4. *Rate of convergence*

We now provide some rate of convergence properties, assuming we have a solution $(\Delta p, \Delta d)$ of (16)–(18). The proofs here follow the outline of [26, Section 3], generalising where necessary to accommodate $\langle \Delta p, \Delta d \rangle \neq 0$, and also to rely less on operator-commutativity. We note that our analysis does not actually depend on the exact form of the linear equations (16)–(17). These conditions merely act as source of proximity to singularities for the whole system, and therefore the analysis could easily be applied to other linear systems sharing (18), arising from optimality conditions for more general classes of problems.

So, let us set

$$p(\alpha) := p + \alpha \Delta p, \quad d(\alpha) := p + \alpha \Delta d, \quad \mu(\alpha) := \operatorname{tr} p(\alpha) \circ d(\alpha)/r. \tag{19}$$

Then, denoting $\Delta := \Delta p \circ \Delta d$,

$$\begin{aligned} r\mu(\alpha) &= \operatorname{tr} p \circ d + \alpha \operatorname{tr}(p \circ \Delta d + d \circ \Delta p) + \alpha^2 \operatorname{tr} \Delta p \circ \Delta d \\ &= r\mu + \alpha(\sigma - 1) r\mu + \alpha^2 \operatorname{tr} \Delta \\ &= (1-\alpha) r\mu + \alpha\sigma r\mu + \alpha^2 \operatorname{tr} \Delta. \end{aligned} \tag{20}$$

The linear constraints of $(p(\alpha), d(\alpha)) \in S_{r\mu(\alpha)}$ obviously automatically continue to hold for any α. The next lemma bounds the non-linear constraints.

Lemma 4.1: *If $(p, d) \in \mathcal{C}^*_\bullet(\gamma)$ for some $\bullet \in \{F, 2, -\infty\}$, then $(p(\alpha), d(\alpha)) \in \mathcal{C}^*_\bullet(\gamma) \cup C_0$ for $\alpha \in [0, \bar{\alpha}]$, where*

$$\bar{\alpha} := \begin{cases} \sigma/\kappa, & \kappa \geq \sigma, \\ 1/(1-\sigma/2), & \kappa = 0, \\ \sqrt{(1-\sigma/2)^2/\kappa^2 + 2/\kappa} - (1-\sigma/2)/\kappa, & 0 \neq \kappa \in (-(1-\sigma/2)^2/2, \sigma), \\ \infty, & \text{otherwise}, \end{cases} \tag{21}$$

and $\kappa := (\|P_e^\perp \Delta\|_F - \gamma \operatorname{tr} \Delta / r)/(\gamma\mu)$. When $\kappa < \sigma$, then $\bar{\alpha} > 1$.

Proof: It suffices to prove that for $\alpha \in (0, \bar{\alpha})$, $\|P_e^\perp(p(\alpha) \circ d(\alpha))\|_\bullet < \gamma\mu(\alpha)$. For, as follows from the relationships presented in Section 4.3, then the same holds for $\bullet = -\infty$, and consequently

$$(1-\gamma)\mu(\alpha) < \min_i \zeta_i(p(\alpha) \circ d(\alpha)) \leq \min_i \zeta_i(Q^{1/2}_{p(\alpha)} d(\alpha)),$$

where the second inequality is proved in [26, Lemma 30], and applies when $p(\alpha) \in \operatorname{int} \mathcal{K}$. But then, taking the power of r on both sides, we get

$$((1-\gamma)\mu(\alpha))^r < \det(Q^{1/2}_{p(\alpha)} d(\alpha)) = \det(p(\alpha)) \det(d(\alpha)),$$

applying [7, Proposition III.4.2] on subalgebras for the equality. Now, by the continuity of the involved quantities in α, this condition would be violated if at some point either $p(\alpha)$ or $d(\alpha)$ reached $\operatorname{bd} \mathcal{K}$ while still $\mu(\alpha) > 0$. But if $\mu(\alpha) = 0$, we must also have $\|P_e^\perp(p(\alpha) \circ d(\alpha))\|_\bullet = 0$, whence $\alpha = \bar{\alpha}$. Thus $(p(\alpha), d(\alpha)) \in C_0$, and we have a solution to the problem.

We have

$$\begin{aligned} P_e^\perp(p(\alpha) \circ d(\alpha)) &= P_e^\perp(p \circ d) + \alpha P_e^\perp(p \circ \Delta d + d \circ \Delta p) + \alpha^2 P_e^\perp(\Delta d \circ \Delta p) \\ &= P_e^\perp(p \circ d) + \alpha P_e^\perp(\sigma\mu e - p \circ d) + \alpha^2 P_e^\perp \Delta \\ &= (1-\alpha) P_e^\perp(p \circ d) + \alpha^2 P_e^\perp \Delta. \end{aligned}$$

To approximate the norm, for $\bullet = F$ we can use the triangle inequality, whereas for $\bullet = 2, -\infty$, we apply [26, Lemma 14], which states that for $x, y \in \mathcal{J}$, $-\min \zeta_i(x + y) \leq -\min \zeta_i(x) + \|y\|_F$, and $\max \zeta_i(x+y) \leq \max \zeta_i(x) + \|y\|_F$. Therefore, for all $\bullet \in \{F, 2, -\infty\}$, we have the approximation

$$\|P_e^\perp(p(\alpha) \circ d(\alpha))\|_\bullet \leq |1-\alpha| \, \|P_e^\perp Q_p^{1/2} d\|_\bullet + \alpha^2 \gamma \|P_e^\perp \Delta\|_F \leq |1-\alpha| \, \gamma\mu + \alpha^2 \|P_e^\perp \Delta\|_F.$$

Comparing this approximation against $\mu(\alpha)$ from (20), we get that

$$\|P_e^\perp(p(\alpha)\circ d(\alpha))\| \leq \gamma\mu(\alpha)$$

if

$$\alpha^2\|P_e^\perp\Delta\|_F \leq (1-\alpha-|1-\alpha|+\alpha\sigma)\gamma\mu + \gamma\alpha^2\operatorname{tr}\Delta/r,$$

i.e., $\alpha^2\kappa \leq (1-\alpha-|1-\alpha|+\alpha\sigma)$.

Suppose we have equality at $0<\alpha\leq 1$. Then $\kappa\geq\sigma$, and we get the bound in (21). On the other hand, if $\kappa<\sigma$, the inequality holds strictly for all $\alpha\in(0,1]$. So equality is reached at $\alpha>1$, and we get the bound in (21) by solving the quadratic equation $\alpha^2\kappa - 2 + \alpha(2-\sigma) = 0$. When $\kappa\neq 0$, there are potentially two solutions,

$$\alpha = \frac{-(1-\sigma/2)\pm\sqrt{(1-\sigma/2)^2+2\kappa}}{\kappa},$$

but the bound in (21) is the one we want. This follows for $\kappa>0$, because the other solution is negative. For $\kappa<0$ this follows from observing that a quadratic function with a negative quadratic term, which is also negative and increasing at $\alpha=0$, has only positive roots, if any. Therefore, the smaller root, if any, gives the bound, and otherwise it is infinite. Solving for the term under the square root to equal zero gives the lower bound for the applicability of the expression in (21). □

Suppose $\operatorname{tr}\Delta > 0$. Then, minimising $\mu(\alpha)$ over $\alpha\geq 0$, we get $\sigma\mu = 2\check{\alpha}\operatorname{tr}\Delta$, or $\check{\alpha} := (1-\sigma)/(2\check{\kappa})$ with $\check{\kappa} := \operatorname{tr}\Delta/(r\mu)$. For convenience, we set $\check{\alpha}=\infty$ when $\operatorname{tr}\Delta\leq 0$.

Lemma 4.2: *Assume the conditions of Lemma 4.1. Let $\hat{\alpha} := \min\{\bar{\alpha},\check{\alpha}\}$. Then*

$$\delta := 1-\mu(\hat{\alpha})/\mu \geq (1-\sigma)\hat{\alpha}/2. \tag{22}$$

Proof: When $\operatorname{tr}\Delta>0$, $\check{\alpha}\geq\alpha$ is equivalent to $\check{\kappa}\alpha\leq(1-\sigma)/2$. Then we find from (20) that

$$\mu(\alpha)/\mu - 1 = (\sigma-1)\alpha + \alpha^2\check{\kappa} \leq (\sigma-1)\alpha + (1/2)(1-\sigma)\alpha = (1/2)(\sigma-1)\alpha.$$

When $\operatorname{tr}\Delta\leq 0$, the same result continues to hold because $\alpha^2\check{\kappa}\leq 0$ may be dropped, and $\sigma-1<0$. Therefore the claim holds when $\check{\alpha}\geq\bar{\alpha}$.

When $\check{\alpha}\leq\bar{\alpha}$, we get that $\mu(\check{\alpha})/\mu - 1 = (\sigma-1)\check{\alpha} + (1-\sigma)\check{\alpha}/2$, which gives the desired result. □

Therefore, to obtain fast decrease in μ, it suffices to bound $\hat{\alpha}$ from below. For, given a lower bound $\hat{\delta}\leq\delta$, a standard argument[1] shows that $\hat{\delta}^{-1}\log\tau^{-1}$ steps are sufficient to ensure that $\mu\leq\tau\overline{\mu}$ for an initial $\overline{\mu}>0$ and desired decrease factor $\tau\in(0,1)$.

If $\kappa<\sigma$, then $\bar{\alpha}>1$ from Lemma 4.1. Therefore in this case, it suffices to have a bound for $\check{\alpha}$ from below. Consequently, it suffices to bound both κ and $\check{\kappa}$ from above. Let us see how far that can be done.

[1]Each step obtains a proportional decrease of at least $1-\hat{\delta}$ in μ, so one obtains the condition $\tau\leq(1-\hat{\delta})^k$. Now apply the approximation $-\log(1-\hat{\delta})\geq\hat{\delta}$.

Lemma 4.3: *Let $u, v \in \mathcal{J}$ and let H_u and H_v be invertible linear operators on $\mathcal{J}$, with the induced norm $\|H\|_F := \max_{x\neq 0} \|Hx\|_F/\|x\|_F$. Then*

$$\|u\|_F\|v\|_F \leq \frac{1}{2}\|H_u^{-1}\|_F\|H_v^{-1}\|_F\big(\|H_u u\|^2 + \|H_v v\|^2\big).$$

Proof: We have $\|u\|_F = \|H_u^{-1}H_u u\| \leq \|H_u^{-1}\|_F\|H_u u\|_F$ and likewise for v. Now apply the inequality $2ab \leq a^2 + b^2$. $\square$

Lemma 4.4: *Suppose $p, d, q = p \circ d \in \operatorname{int}\mathcal{K}$, and that* (18) *holds. Suppose H_0 is an invertible linear operator in $\mathcal{J}$ that satisfies $H_0 q = q^{1/2}$ and $H_0 e = q^{-1/2}$. Let $H_d := H_0 L(p)$ and $H_p := H_0 L(d)$. Then $\|H_d\Delta d\|_F^2 + \|H_p\Delta p\|_F^2 = \theta - 2\langle H_p\Delta d, H_d\Delta d\rangle$ with*

$$\theta := \theta(q,\sigma) := \sum_{i=1}^{r} \frac{(\sigma\mu(q) - \zeta_i(q))^2}{\zeta_i(q)}.$$

Proof: Multiplying (18) from the left by H_0, we get

$$H_d\Delta d + H_p\Delta p = H_0(\sigma\mu e - p\circ d) = \sigma\mu q^{-1/2} - q^{1/2},$$

where $\|\sigma\mu q^{-1/2} - q^{1/2}\|^2 = \operatorname{tr}[(\sigma\mu q^{-1/2} - q^{1/2})^2] = \theta$. On the other hand,

$$\|H_d\Delta d + H_p\Delta p\|_F^2 - 2\langle H_d\Delta d, H_p\Delta p\rangle = \|H_d\Delta d\|_F^2 + \|H_p\Delta p\|_F^2. \quad \square$$

Combining Lemmas 4.3 and 4.4, we get the bound

$$\|\Delta p\|_F\|\Delta d\|_F \leq \frac{1}{2}\|H_p^{-1}\|_F\|H_d^{-1}\|_F(\theta - 2\langle H_p\Delta p, H_d\Delta d\rangle).$$

Now, if $\langle H_p\Delta p, H_d\Delta d\rangle \geq 0$, we may drop it. Otherwise, we have for $\beta = 1$ that

$$-\langle H_p\Delta p, H_d\Delta d\rangle \leq \beta\|H_p\Delta p\|_F\|H_d\Delta d\|_F \leq \frac{\beta}{2}(\|H_p\Delta p\|_F^2 + \|H_d\Delta d\|_F^2).$$

If we can actually take $\beta < 1$, we get a geometrical series converging to the limit $(\|H_p^{-1}\|_F\|H_d^{-1}\|_F/2)\theta/(1-\beta)$, and thus the estimate

$$\|\Delta p\|_F\|\Delta d\|_F \leq \frac{\|H_p^{-1}\|_F\|H_d^{-1}\|_F\theta}{2(1-\beta)}.$$

On the other hand, if $\beta = 1$ is the only option, we have $-\langle H_p\Delta p, H_d\Delta d\rangle = \|H_p\Delta p\|_F\|H_d\Delta d\|_F$, which says that $H_0L(d)\Delta p + \tau H_0L(p)\Delta d = 0$ for some $\tau \geq 0$. That is, $L(d)\Delta p + \tau L(p)\Delta d = 0$, which means (16)–(18) must be singular. Consequently, if $\beta \nearrow 1$, (p, d) must be approaching a singularity of the system. Sufficiently far from a singularity, we thus get the following bounds.

Lemma 4.5: *Suppose $\beta \in [0, 1)$ and*

$$-\langle H_p\Delta p, H_d\Delta d\rangle \leq \beta\|H_p\Delta p\|_F\|H_d\Delta d\|_F.$$

Then

$$\kappa \leq (1/\gamma + 1/r)\theta' \quad \text{and} \quad \breve{\kappa} \leq (1/r)\theta'$$

for

$$\theta' := \frac{\|H_p^{-1}\|_F \|H_d^{-1}\|_F}{2(1-\beta)\mu}\theta.$$

Consequently

$$\delta^{-1} \leq 2\max\Big\{\frac{1/\gamma + 1/r}{\sigma(1-\sigma)}\theta', \frac{2/r}{(1-\sigma)^2}\theta', \frac{1}{1-\sigma}\Big\}.$$

Proof: Note that we have both $\|\Delta\|_F \leq \|\Delta d\|_F \|\Delta p\|_F$, as remarked in Section 2.2, as well as $-\operatorname{tr}\Delta \leq \|\Delta d\|_F \|\Delta p\|_F$. Thus $\kappa \leq (1+\gamma/r)\|\Delta d\|_F\|\Delta p\|_F/(\gamma\mu)$ and $\breve{\kappa} \leq$. Approximating

$$\|\Delta d\|_F \|\Delta p\|_F/\mu \leq \theta',$$

as discussed above, and noting that $(1+\gamma/r)/\gamma = 1/\gamma + 1/r$, yields the claimed bounds for κ and $\breve{\kappa}$. Now apply these bounds in $\bar{\alpha}^{-1} = \kappa/\sigma$ $(\kappa \geq \sigma)$ and $\breve{\alpha}^{-1} = 2\breve{\kappa}/(1-\sigma)$, and insert the results into (22), i.e., $\delta^{-1} \leq 2\hat{\alpha}^{-1}/(1-\sigma)$, to yield the first two terms of the maximum expression. The last term is obtained by bounding $\hat{\alpha} \leq \bar{\alpha} \leq 1$. □

The following result ensures that θ/μ stays bounded in the neighbourhoods $\mathcal{C}_\bullet$ under consideration.

Lemma 4.6: *Suppose* $\|P_e^\perp w\|_\bullet \leq \gamma\mu(w)$ *for* $\gamma \in (0,1)$, $w \in \mathcal{J}$. *Then, for* $\sigma > 0$,

$$\theta(w,\sigma) \leq \Big(\frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma}\Big)\mu(w) \quad \textit{when} \quad \bullet = F, \textit{ and} \tag{23}$$

$$\theta(w,\sigma) \leq \Big(1 - 2\sigma + \frac{\sigma^2}{1-\gamma}\Big)\mu(w)r \quad \textit{when} \quad \bullet = 2, -\infty. \tag{24}$$

Proof: See the proof of [26, Lemma 35], that actually only depends on the properties of w, not of s and x (p and d). □

It remains to consider H_p and H_d.

Lemma 4.7: *Suppose* $p, d, q \in \operatorname{int}\mathcal{K}$. *Then in Lemma 4.4,*

(i) We may take $H_0 = L(q)^{-1/2}$ *or* $H_0 = L(q^{-1/2})$.
(ii) When p *and* d *operator-commute, we may take* $H_0 = L(d)^{-1/2}L(p)^{-1/2}$, *and get* $\|H_p^{-1}\|_F\|H_d^{-1}\|_F \leq \sqrt{\operatorname{cond}(H)}$ *for* $H := L(d)^{-1}L(p)$.

Proof: (i) Clearly the operators are invertible. Furthermore, $L(q)^{-1/2} = L(q^{-1/2})$ on the space spanned by the eigenvectors of q. Therefore, for both alternatives, $H_0 q = q^{1/2}$ and $H_0 e = q^{-1/2}$.

(ii) Since $p, d \in \operatorname{int}\mathcal{K}$ operator-commute, H_0 is symmetric and they share a Jordan frame, wherefore $q^t = p^t \circ d^t$. Thus $H_0 q = q^{1/2}$ and $H_0 e = q^{-1/2}$. Also by operator-commutativity $H_d = H_0 L(p) = H^{1/2}$ and $H_p = H_0 L(d) = H^{-1/2}$, so that $\|H_p^{-1}\|_F\|H_d^{-1}\|_F = (\|H\|_F\|H^{-1}\|_F)^{1/2} = \sqrt{\operatorname{cond}(H)}$. □

The results of this section are summarised in the following algorithm and theorem, recalling that we may scale our representation of f_ν. For $\bullet = F$, better $\sqrt{r}$ complexities could actually be obtained by limiting σ; see [26].

Algorithm 4.1: Interior point method for DC problems on symmetric cones

(1) Choose target accuracy $\underline{\mu} > 0$, parameters $\gamma, \sigma \in (0,1)$, and an initial iterate $(p,d) \in \mathcal{C}_\bullet(\gamma) \cap G^{-1}_{r\mu}(y,0)$ for some $\bullet \in \{F, 2, -\infty\}$ and $y \in \mathbb{R}^m$.
(2) Choose a scaling Q_v such that $(\widetilde{p}, \underset{\sim}{d}) \in \mathcal{C}^*_\bullet(\gamma)$, and a H_0 satisfying the constraints of Lemma 4.4 wrt. $(\widetilde{p}, \underset{\sim}{d})$.
(3) Solve $(\Delta\widetilde{p}, \Delta\underset{\sim}{d})$ from (16)–(18) if possible. Otherwise stop with failure.
(4) Update $(p,d) := (Q_v^{-1}\widetilde{p}(\hat{\alpha}), Q_v\underset{\sim}{d}(\hat{\alpha}))$ as the new iterate.
(5) If $\mu \leq \underline{\mu}$, stop. Otherwise continue from Step (2)

Theorem 4.8 : *Suppose that Step (3) of Algorithm 4.1 always succeeds, and there exists at each iteration an H_0 satisfying the conditions of Lemma 4.4 wrt. $(\widetilde{p}, \underset{\sim}{d})$. Suppose furthermore that $\|H_p^{-1}\|_F\|H_d^{-1}\|_F/(1-\beta)$ can be bounded from above by a constant $M < \infty$. Denote by $\overline{\mu}$ the initial (maximal) μ and let $\tau := \underline{\mu}/\overline{\mu}$. Then $O(Mr\log\tau^{-1})$ iterations are sufficient for $\mu \leq \underline{\mu}$.*

Proof: Note that since $\mathcal{C}^*_\bullet(\gamma) \subset \mathcal{C}_\bullet(\gamma)$, and the latter is scaling invariant, after reverse scaling still $(Q_v^{-1}\widetilde{p}(\alpha), Q_v\underset{\sim}{d}(\alpha)) \in \mathcal{C}_\bullet(\gamma)$. Therefore Step (4) and the method are well-defined.

Other dependencies on r in the bound for δ^{-1} from Lemma 4.5 can be approximated away, except the linear one in (23) or (24). Thus $\delta^{-1} = O(Mr)$, and the claim follows from the discussion following Lemma 4.2. □

4.5. *Operator-commutative scalings*

Suppose we choose the scaling such that $\widetilde{p} = Q_v p$ and $\underset{\sim}{d} = Q_v^{-1} d$ operator-commute. As discussed in Section 4.3, then $(p,d) \in \mathcal{C}_\bullet(\gamma)$ implies $(\widetilde{p}, \underset{\sim}{d}) \in \mathcal{C}^*_\bullet(\gamma)$, taking care of that assumption in Theorem 4.8. Lemma 4.7 then says that it remains to bound $\mathrm{cond}(H)$ (and stay away from a singularity).

In the Nesterov-Todd method [20], the scaling element is chosen to be v for the unique element for which $Q_{v^2}p = d$, expressible as $v = (Q_{p^{1/2}}(Q_{p^{1/2}}d)^{-1/2})^{-1/2}$; see [26]. Then $\widetilde{p} = \underset{\sim}{d}$ operator-commute, and $L(\underset{\sim}{d})^{-1}L(\widetilde{p}) = I$, so that consequently $\mathrm{cond}(H) = 1$. In the so-called "xs" method, $v = d^{1/2}$, so that $\underset{\sim}{d} = e$, wherefore we have operator-commutativity, and get $\mathrm{cond}(H) \leq 2/(1-\gamma)$ for $\bullet = 2, F$, and $\mathrm{cond}(H) \leq r/(1-\gamma)$ for $\bullet = -\infty$ [26]. In the "sx" method $v = p^{-1/2}$, with similar results. More generally, the so-called power class of scalings (or search directions) considered by [19], yield bounded $\mathrm{cond}(H)$.

Of course, the question remains: what is the effect of scaling on the closeness of the system (16)–(18) to a singularity? It follows from Lemma 3.8 that this is at least somewhat unaffected close to a central selection.

5. Sequential convex programming

We now consider a simple method for general DC functions f_ν, and the application of interior point methods to a sub-problem when f is of the form (3).

5.1. *The general idea*

Consider two arbitrary finite convex functions f and ν on $\mathbb{R}^m$. Let $\underline{\epsilon} \geq 2\rho \geq 0$ be chosen. Suppose $z \in \partial_\rho\nu(y)$, $z \notin \partial_{\underline{\epsilon}-\rho}f(y)$. In other words,

$$\nu(y') - \nu(y) \geq z^T(y'-y) - \rho, \qquad \text{for all } y', \tag{25}$$

$$f(y'') - f(y) < z^T(y''-y) - (\underline{\epsilon} - \rho), \qquad \text{for some } y''. \tag{26}$$

Setting $y' = y''$ and summing,

$$f_\nu(y'') - f_\nu(y) < -\underline{\epsilon} + 2\rho$$

so that y is not $\underline{\epsilon} - 2\rho$ -optimal.

Suppose then that we have $z \in \partial_{\epsilon'} f(\hat{y})$, i.e.

$$f(y') - f(\hat{y}) \geq z^T(y' - \hat{y}) - \epsilon', \qquad \text{for all } y'.$$

Setting $y' = y''$, and summing with (26), we have

$$f(y) - f(\hat{y}) > z^T(y - \hat{y}) - \epsilon' + (\underline{\epsilon} - \rho).$$

Setting $y' = \hat{y}$ and further summing with (25),

$$f_\nu(y) - f_\nu(\hat{y}) > (\underline{\epsilon} - 2\rho) - \epsilon'. \tag{27}$$

Thus, if $\epsilon' \leq \sigma_{\text{SCP}}(\underline{\epsilon} - 2\rho)$ for $\sigma_{\text{SCP}} \in (0, 1)$, a reduction of $(1 - \sigma_{\text{SCP}})(\underline{\epsilon} - 2\rho)$ has been achieved in the value of f_ν.

The conceptual algorithm for finding $\underline{\epsilon}$-semi-critical points of f_ν is now clear.

Algorithm 5.1: Sequential convex programming (SCP) method

(1) Choose target accuracy $\underline{\epsilon} > 0$, gradient accuracy $\rho \in [0, \underline{\epsilon}/2)$, stepwise reduction $\sigma_{\text{SCP}} \in (0, 1)$, and an initial iterate $y_{[0]}$.
(2) Select a subgradient $z_{[k]} \in \partial_\rho \nu(y_{[k]})$.
(3) Set $\epsilon' := \sigma_{\text{SCP}}(\underline{\epsilon} - 2\rho)$, and find $\hat{y}$ such that $z_{[k]} \in \partial_{\epsilon'} f(\hat{y})$.
(4) If a reduction of $(1 - \sigma_{\text{SCP}})(\underline{\epsilon} - 2\rho)$ is not obtained in the value of f_ν, by the above analysis it must have been that $z_{[k]} \in \partial_{\underline{\epsilon}-\rho} f(y_{[k]})$, so that $0 \in \partial^{\text{DC}}_{\underline{\epsilon}} f_\nu(y_{[k]})$, and we already were at a $\underline{\epsilon}$-semi-critical point. Therefore, stop with result $y_{[k]}$.
(5) Otherwise repeat from Step (2) with $y_{[k+1]} := \hat{y}$, and $k := k + 1$.

Clearly, as a constant reduction in the value of f_ν is achieved on each non-final iteration, the method is convergent if Step (3) always succeeds, and f_ν is bounded from below. For the success, we should have $\mathcal{R}(\partial\nu) \subset \mathcal{R}(\partial f)$. The stricter bound $\operatorname{cl} \mathcal{R}(\partial\nu) \subset \operatorname{int} \mathcal{R}(\partial f)$ along with bounded $\mathcal{R}(\partial\nu)$ in fact ensures that f_ν has bounded level sets and is therefore bounded from below [30, Corollary 3].

We note that this method can be seen as an approximate variant of DCA [2], the "simplified" version of which amounts to $\rho = \underline{\epsilon} = 0$ (while the "complete" version sets further restrictions). The method of truncated codifferential considered by Demyanov et al. [6] also bears many parallels to SCP.

Remark 5.1: Alternatively, instead of fixing ϵ' in Step (3), we may attempt to find $\hat{y}$ and $\epsilon' > 0$ with $z \in \partial_{\epsilon'} f(\hat{y})$, such that the objective function value is reduced by $0 < \Delta_{[k]} \leq \underline{\epsilon} - 2\rho$, or (27) is violated (for $y = y_{[k]}$), one of which must occur for small enough $\epsilon' > 0$.

Remark 5.2: The SCP argument actually proves convergence for inexact K-means -style local convex optimisation methods. Suppose $f_\nu(y) = f(y) - \nu(y)$ for $\nu(y) := \max_{t \in T} \nu_t(y)$ for some finite index set T and convex functions ν_t, and that $f_t := f - \nu_t$ are convex. Now, suppose $f_\nu(y) = f_t(y)$, and choose in the SCP method, f_t for f, 0 for ν, $z = 0$ and $\rho = 0$. If the predicted decrease is not achieved, then the SCP argument says $0 \in \partial_{\underline{\epsilon}} f_t(y)$, that is $f(y') - f(y) \geq \nu_t(y') - \nu_t(y) - \underline{\epsilon}$ for all

y'. But then for any $z \in \partial\nu_t(y) \subset \partial\nu(y)$, $f(y') - f(y) \geq z^T(y' - y) - \underline{\epsilon}$. This says $z \in \partial_{\underline{\epsilon}} f(y)$, so that y is $\underline{\epsilon}$-semi-critical for f_ν.

5.2. *Interior point SCP*

If f (but not necessarily ν) has the form (3), we may apply Algorithm 4.1 in Step (3) of Algorithm 5.1 to reducing $\epsilon' > 0$ in $z \in \partial_{\epsilon'} f(\hat{y})$, after finding initial values for which this holds. For, as is clear from the analysis, Algorithm 4.1 always maintains the linear constraints for any set values, and therefore works for other values besides $z = 0$. If we can initialise each iteration in a bounded manner, we have finite convergence. More precisely,

Theorem 5.1: *Suppose that for all $y_{[k]}$ and $z_{[k]} \in \partial_\rho \nu(y_{[k]})$, we can (in negligible time) initialise $(p_f, d_f) \in G^{-1}_{f,\bar{\epsilon}}(y', z_{[k]}) \cap \mathcal{C}_\bullet(\gamma)$ at some y' for fixed $\bar{\epsilon} \geq f_\nu(y_{[0]}) - \min f_\nu$, $\gamma \in (0, 1)$, and $\bullet \in \{F, 2, -\infty\}$. Then, if Algorithm 4.1 is used in Step (3) with one of the operator-commutative scalings from Section 4.5, $O(K_{\gamma,r_f} \tau^{-1} \log \tau^{-1})$ steps of the interior point method are sufficient to reach an $\underline{\epsilon}$-semi-critical point, with $\tau := (\underline{\epsilon} - 2\rho)/\bar{\epsilon}$, and K_{γ,r_f} a polynomial of $1/(1-\gamma)$ and r_f,*

Here and in the rest of this section, $G^{-1}_{f,\epsilon}$ is G^{-1}_ϵ as defined in (8) for the data of f, while without the specifier, the data of all of f_ν is implied, as before. $\mathcal{C}_\bullet(\gamma)$ is a subset of one of $\mathcal{K}$, $\mathcal{K}_f$, or $\mathcal{K}_\nu$, depending on the context.

The factor K_{γ,r_f} replaces Mr_f and omitted terms from Theorem 4.8, where the dependence on γ was de-emphasised, being something that can be chosen arbitrarily small by suitable initialisation. Here, however, z limits the quality of the initialisation – which cannot be done if $z \notin \mathcal{R}(\partial f)$.

Proof: The term $1/(1-\gamma)$ is the dominant one involving γ as $\gamma \nearrow 1$ in the bounds of Lemma 4.6 and the bounds for $\mathrm{cond}(H)$ in Section 4.5. Therefore, similarly to the proof of Theorem 4.8, we find from Lemmas 4.5 and 4.2 that to find an $\underline{\epsilon} - 2\rho$ critical point, each invocation of Step (3) requires $O(K_{\gamma,r_f} \log \tau^{-1})$ steps of the interior point method, where K_{γ,r_f} is as claimed.

Since each non-terminal step of the SCP algorithm achieves a reduction of at least $(1 - \sigma_{\mathrm{SCP}})(\underline{\epsilon} - 2\rho)$ in the value of f_ν, and $\Delta_0 := f_\nu(y_{[0]}) - \min f_\nu \leq \bar{\epsilon}$, we get that $n \geq \Delta_0/((1 - \sigma_{\mathrm{SCP}})(\underline{\epsilon} - 2\rho)) = O(\tau^{-1})$ iterations of the SCP method are sufficient. This results in the claimed total number of iterations of the interior point method. □

Next we study when the initialisation required above can be performed, and with what quality. We begin with a few basic results needed towards that end.

Lemma 5.2: *Suppose f_ν is bounded from below, $\rho \geq 0$, $\Delta_0 \geq f_\nu(y) - \min f_\nu$, and $z \in \partial_\rho \nu(y)$. Then $z \in \partial_{\Delta_0+\rho} f(y)$.*

Proof: By assumption $\rho \geq \nu(y) - \nu(y') + z^T(y' - y)$ and $\Delta_0 \geq f(y) - \nu(y) - f(y') + \nu(y')$ for all y'. By combining these inequalities, we get the claim. □

Lemma 5.2 and (7) thus show the existence of some $(p_f, d_f) \in G^{-1}_{f,\Delta_0+\rho}(y_{[k]}, z_{[k]})$. The objective is then to improve $(p_f, d_f) \in \mathcal{C}_\bullet(\gamma)$ for fixed $\gamma \in (0, 1)$ without $\bar{\epsilon} \geq \Delta_0 + \rho$ increasing unboundedly.

To provide such results, we need to show that $\|\cdot\|_{-\infty}$ actually satisfies the triangle inequality (although it is not a norm).

Lemma 5.3: *Suppose $x, y \in \mathcal{J}$. Then $\|x + y\|_{-\infty} \leq \|x\|_{-\infty} + \|y\|_{-\infty}$.*

Proof: As defined in Section 4.3, $\|z\|_{-\infty} = -\min_i \zeta_i(z)$, so it suffices to show $\min_i \zeta_i(z) \geq \min_i \zeta_i(x) + \min_i \zeta_i(y)$ for $z = x + y$. Let $x = \sum_{i=1}^r \zeta_i(x) x_i$, $y = \sum_{i=1}^r \zeta_i(y) y_i$, and $z = \sum_{i=1}^r \zeta_i(z) z_i$ be the decompositions of $x, y, z \in \mathcal{K}$ into sums of primitive idempotents. Since primitive idempotents are in $\mathcal{K}$, their inner product is non-negative. Applying $\sum_j x_j = e$ and $\operatorname{tr} z_i = 1$, it thus follows that

$$\begin{aligned}\zeta_i(z) = \langle z_i, z\rangle = \langle z_i, x\rangle + \langle z_i, y\rangle &= \sum_j (\zeta_j(x)\langle z_i, x_j\rangle + \zeta_j(y)\langle z_i, y_j\rangle) \\ &\geq \min_k \zeta_k(x)\langle z_i, \sum_j x_j\rangle + \min_k \zeta_k(y)\langle z_i, \sum_j y_j\rangle = \min_j \zeta_j(x) + \min_j \zeta_j(y). \quad \square\end{aligned}$$

Assumption 5.4: We assume that $A(p_1, \ldots, p_n) = (\langle a_1', p_1\rangle, \ldots, \langle a_n', p_n\rangle)$ as in Assumption 3.9, along with $(\mathcal{R}(A^*) \cap \operatorname{int} \mathcal{K})^{-1} \subset \mathcal{N}(B) \cap \mathcal{N}(\langle c, \cdot\rangle)$.

Example 5.5 This assumption is satisfied by combinations of Euclidean norms (cf. Example 3.1), where of $p_i = (p_i^0, \bar{p}_i) \in \mathcal{E}_{m+1}$, A depends only on p_i^0, and B and $\langle c, \cdot\rangle$ on $\bar{p}_i$.

When the assumption holds, we set $a := (\phi_1 a_1', \ldots, \phi_n a_n')$, where $\phi_i \in \mathbb{R}$ is chosen so that $Aa^{-1} = b$, i.e. $r_i = \phi_i b_i$. Then $a \in \mathcal{R}(A^*) \cap \operatorname{int} \mathcal{K}$, so that $\langle a^{-1}, B^* y\rangle = \langle a^{-1}, c\rangle = 0$. Also, $\mu(a \circ p) = 1$ for any $p \in V = \{p \in \mathcal{K} \mid Ap = b\}$, because $\langle a, p\rangle = \sum_i \phi_i \langle a_i', p_{f,i}\rangle = \sum_i \phi_i b_i = \sum_i r_i$.

Lemma 5.6: *Suppose Assumption 5.4 holds, and that* $(p', d') \in G_{\epsilon'}^{-1}(y, z) \cap \mathcal{C}_{-\infty}(\gamma')$. *Then, for* $0 < \psi < \gamma \leq \gamma'$, *there exist* $(p, d) \in G_\epsilon^{-1}(y, \psi z) \cap \mathcal{C}_{-\infty}(\gamma)$ *with*

$$\epsilon := \psi \frac{\gamma' - \psi}{\gamma - \psi}\langle p', d'\rangle + \frac{(1-\psi)^2}{\gamma - \psi}\langle a^{-1}, d'\rangle \leq \frac{1 + (\gamma' - 2)\psi}{\gamma - \psi}\epsilon' + \frac{(1-\psi)^2}{\gamma - \psi} v(y), \tag{28}$$

where $v(y) := \sup_{p \in V}\langle p, B^* y + c\rangle$. *In particular, with* $\gamma' = 1$ *and* $\gamma = (1 + \psi)/2$, *we get* $\epsilon = 2\langle p, d'\rangle$ *and* $1/(1-\gamma) = O(1/(1-\psi))$.

By the definition of f and ν, when the lemma is applied to the data of f alone, $v = f$, and when it is applied to f_ν, $v = f + \nu$.

Proof: Letting $p := \psi p' + (1-\psi)a^{-1}$, we have $Bp = \psi z$, and by convexity $p \in V$. Furthermore, $Q_a^{1/2} p = Q_a^{1/2}(\psi p') + (1-\psi)e$, so that $P_e^\perp Q_a^{1/2} p = \psi P_e^\perp Q_a^{1/2} p'$. Since $Q_a^{1/2} p' \in \mathcal{K}$, we have $\min_i \zeta_i(Q_a^{1/2} p') \geq 0$, and then

$$\|P_e^\perp Q_a^{1/2} p\|_{-\infty} = \psi \|P_e^\perp Q_a^{1/2} p'\|_{-\infty} \leq \psi \mu(Q_a^{1/2} p') = \psi = \psi \mu(a \circ p). \tag{29}$$

Now, let $d := d' + \lambda a$, for yet unspecified $\lambda \geq 0$. Clearly $d \in \mathcal{K}$. Now $Q_p^{1/2} d = \lambda Q_p^{1/2} a + Q_p^{1/2} d'$, and both of the components are in $\mathcal{K}$. Therefore, we may apply Lemma 5.3 and get by the symmetricity $\|P_e^\perp Q_p^{1/2} a\|_{-\infty} = \|P_e^\perp Q_a^{1/2} p\|_{-\infty}$ [26, Proposition 21] that

$$\begin{aligned}\|P_e^\perp Q_p^{1/2} d\|_{-\infty} &\leq \lambda \|P_e^\perp Q_p^{1/2} a\|_{-\infty} + \|P_e^\perp Q_p^{1/2} d'\|_{-\infty} \\ &\leq \lambda \|P_e^\perp Q_p^{1/2} a\|_{-\infty} + \psi \|P_e^\perp Q_{d'}^{1/2} p'\|_{-\infty} + (1-\psi)\|P_e^\perp Q_{d'}^{1/2} a^{-1}\|_{-\infty} \\ &\leq \lambda\psi + \psi\gamma' \mu(p' \circ d') + (1-\psi)\mu(a^{-1} \circ d').\end{aligned}$$

Since

$$\mu(p \circ d) = \lambda\mu(p \circ a) + \mu(p \circ d') = \lambda + \psi\mu(p' \circ d') + (1-\psi)\mu(a^{-1} \circ d'), \tag{30}$$

we therefore have $\|P_e^{\perp} Q_p^{1/2} d\|_{-\infty} \leq \gamma\mu(d \circ p)$, if

$$\psi(\gamma' - \gamma)\mu(p' \circ d') + (1-\psi)(1-\gamma)\mu(a^{-1} \circ d') \leq (\gamma - \psi)\lambda.$$

Setting this to equality and inserting the resulting λ in (30), gives the first half of (28) (as $\epsilon = r\mu(p \circ d)$).

For the second half of (28), observe that $\langle a^{-1}, d'\rangle = \langle p', d'\rangle + \langle a^{-1} - p', d'\rangle = \langle p', d'\rangle + \langle p' - a^{-1}, B^*y + c\rangle = \langle p', d'\rangle + \langle p', B^*y + c\rangle \leq \epsilon + v(y)$ by Assumption 5.4.

Finally, setting $\gamma' = 1$ and $\gamma = (1+\psi)/2$, we have $\gamma - \psi = (1-\psi)/2$, and therefore $\epsilon = 2(\psi\langle p', d'\rangle + (1-\psi)\langle a^{-1}, d'\rangle)$. By the definition of p, this proves the claim for that case. □

Lemma 5.7: *Suppose Assumption 5.4 holds for f, and that $\mathcal{R}(\partial\nu) \subset \psi\mathcal{R}(\partial f)$ for some $\psi \in (0,1)$. Then there exist $(p_f, d_f) \in G^{-1}_{f,\bar\epsilon}(y,z) \cap \mathcal{C}_{-\infty}(\gamma)$, $\gamma \in (0,1)$, with $1/(1-\gamma) = O(1/(1-\psi))$, in the following cases:*

(i) Varying y with $f_\nu(y) - \min f_\nu \leq \Delta_0$ and $z \in \partial_\rho\nu(y)$, in which case $\bar\epsilon = O(\Delta_0 + \rho + \|V_f\|_F\|c_f\|_F)$.

(ii) Fixed y with $z \in \mathcal{R}(\partial\nu)$, in which case $\bar\epsilon = O(f(y)) = O(\|V_f\|_F\|B_f^* y + c_f\|_F)$.

As usual, the set norm is defined as $\|V_f\|_F := \max_{p\in V_f} \|p\|_F$.

Proof: Note that $\operatorname{cl}\mathcal{R}(\partial f) = \operatorname{cl}\bigcup_{y\in\mathbb{R}^m,\epsilon\geq 0} \partial_\epsilon f(y) = B_f V_f$, also from the expression (7). Therefore, for $z \in \mathcal{R}(\partial\nu)$, there exists $p' \in V_f$ such that $B_f p' = z/\psi$. An application of Lemma 5.6 to (p', d') and z/ψ with $\gamma' = 1$ and $\gamma = (1+\psi)/2$ then provides $(p_f, d_f) := (p, d)$ and the requested bounds as follows:

(i) Let $(p'', d') \in G^{-1}_{f,\Delta_0+\rho}(y,z)$ as shown to exist by Lemma 5.2 and the representation (7). Now, for the p provided by Lemma 5.6 we approximate $\langle p, d'\rangle = \langle p'', d'\rangle + \langle p - p'', d'\rangle = \langle p'', d'\rangle + \langle p'' - p, c_f\rangle \leq \Delta_0 + \rho + 2\|V_f\|_F\|c_f\|_F$, where in the second equality we have used $B_f p = B_f p'' = z$ and $A_f p = A_f p'' = b$.

(ii) Choose $(p'', d') \in G^{-1}_{f,0}(y, z')$ for some $z' \in \partial f(y)$. Then, as in case i), $\langle p, d'\rangle = \langle p'', d'\rangle + \langle p - p'', d'\rangle = \langle p'' - p, B_f^* y + c_f\rangle$, and we readily get the claim by the definition of f. □

According to Lemma 5.7, there then is a solution to our initialisation problem under rather reasonable assumptions; cf. the level-boundedness results of [30, Theorem 7]. But when can we actually find p such that $B_f p = z/\psi$ in V_f? Since $\operatorname{tr}(Q_a^{1/2} p)$ is constant, by the proof of Lemma 5.6, $\|Q_a^{1/2} p\|_{-\infty}$ can be made small enough to imply that $p \in \mathcal{K}$. Therefore, after scaling by a to work on $\widetilde{p} := Q_a^{1/2} p$, and relaxing the norm to $\bullet \in \{F, 2, -\infty\}$, this problem may be cast as $\min_{\widetilde{p}} \|\widetilde{p}\|_\bullet$ subject to $W\widetilde{p} = x_\psi$ and $\widetilde{p} \in \mathcal{K}$, where $W\widetilde{p} := (A_f Q_a^{-1/2}\widetilde{p}, B_f Q_a^{-1/2}\widetilde{p})$ and $x_\psi := (b, z/\psi)$.

If $\bullet = -\infty$, there exists an interior solution for non-minimal ψ. The problem then becomes $\min_{\widetilde{p}}(-\min_j \zeta_j(\widetilde{p})) = \min_{\widetilde{p}} \max_j(-\zeta_j(\widetilde{p}))$. If f has the product presentation of Assumption 5.4, and each of the cones $\mathcal{K}_i$ are second-order cones, the smallest eigenvalue in each cone is $\widetilde{p}_i^0 - \|\bar{\widetilde{p}}_i\|$. But $\widetilde{p}_i^0$ is fixed because $b = A_f Q_a^{-1/2}\widetilde{p} = (\phi_1^{-1}\langle e, \widetilde{p}_1\rangle, \ldots, \phi_n^{-1}\langle e, \widetilde{p}_2\rangle) = (b_1\widetilde{p}_1^0, \ldots, b_n\widetilde{p}_n^0)$. Therefore the problem becomes $\min_{\widetilde{p}} \max_i \|\bar{\widetilde{p}}_i\|$ subject to the linear constraints.

If we set $\bullet = F$, we have $\widetilde{p} = W^\dagger x_\psi$ for the Moore-Penrose pseudo-inverse

$W^\dagger = W^*(WW^*)^{-1}$ (as by assumption $\mathcal{N}(W^*) = \{0\}$), if the minimiser $\widetilde{p} \in \operatorname{int} \mathcal{K}$. Unfortunately this may not be so, unless the norm is small enough that there actually exists a solution $(p, a) \in \mathcal{C}_F(1)$. In some applications the pseudo-inverse however provides a usable result (and is the solution for $\bullet = -\infty$ as well, in fact).

Example 5.8 In the simple case of the spatial median in $\mathbb{R}^m$, $f(y) = \sum_{i=1}^n \|y - a_i\|$, as in general for sums of Euclidean norms, we have $p_f = (p_{f,1}, \ldots, p_{f,n})$ with $p_{f,i} = (p_{f,i}^0, \bar{p}_{f,i}) \in \mathcal{E}^{m+1}$, and $a_i' = e$. Furthermore, $B_f p_f = \sum_i \bar{p}_{f,i}$, so that a simple solution with $p_{f,i} = p_{f,j}$ exists, when at all $z \in B_f V_f$. This extends to sums of spatial medians ($\sum_k f(y_k)$), and suffices for our forthcoming application examples, where $\mathcal{R}(\partial\nu)$ is small enough to be covered by the spatial median component of f, whence we may take $\bar{p}_{f,i} = 0$ for any remaining terms.

Remark 5.3: In the SCP method and case i) of Lemma 5.7, actually $\bar{\epsilon} = O(\Delta_0 + \|V_f\|_F \|c_f\|_F)$. This is because, if $f_\nu(y) - \min f_\nu \le 2\rho \le \underline{\epsilon}$, then choosing $z \in \partial\nu(y)$, we have $z \in \partial_{\underline{\epsilon}} f(y)$, by Lemma 5.2, so y is $\underline{\epsilon}$-semi-critical. Therefore, when y is not $\underline{\epsilon}$-semi-critical, we have $\Delta_0 \ge f_\nu(y) - \min f_\nu \ge 2\rho$, yielding the claim.

6. A filter method

6.1. *The idea*

The idea of the filter method was first introduced for constrained optimisation by Fletcher and Leyffer [12] in a sequential quadratic programming (SQP) framework, with convergence proved in [11], for the case considered. Other works in filter algorithms that seem most related to our work include [27, 31], where interior point approaches are considered.

The filter is basically a multi-dimensional generalisation of a monotonically decreasing sequence bounded from below, where the decrease at each step is sufficient by some criterion. Each point inserted in the filter defines a cone of other points it dominates. Points belonging in an envelope of such a cone are not allowed in future iterations. A filter method is therefore multi-objective optimisation applied to single-objective problems, where typically the additional objectives are related to the constraints of the problem.

In practical methods in the literature so far, there are only two objectives, and each of them are improved separately. One of them, typically the original objective function value, is assigned to be the primary objective, and decrease in it is sought while allowed by the filter, and some additional sufficient decrease conditions are met. New points are inserted in the filter at appropriate places, to force convergence in the future. When this primary phase of the algorithm runs into trouble, a *restoration phase* is entered, with the purpose of improving the second objective and restoring feasibility and acceptability to the filter. Often this restoration method is taken to be a black box.

The restoration method in [27], however, is closely related to the primary method, and merely advances slightly differently. Indeed, although rather general (C^2) constrained nonlinear programming is considered therein, the resulting analysis bears many parallels to the work in Section 4, and more generally the work on linear programming on symmetric cones. Their two elements of the filter actually include the values $\mu(p \circ d)$ and $\|P_e^\perp(p \circ d)\|$ (in the non-negative orthant of $\mathbb{R}^m$, instead of general symmetric cones), plus additional terms related to dissatisfaction of linear constraints. However, to prove convergence for the filter method, it is assumed that the equivalent of the system (16)–(18) is suitably far from a singularity. But with such assumptions, the methods of Section 4 do already converge, “fast”. It is our intent to use the idea of the filter method to circumvent that assumption. We will use a filter and a restoration method to restore feasibility, when the main interior point method runs into trouble. To do this, we apply the results of Section 5, as a consequence of which our restoration phase algorithm will also be closely related to the primary phase algorithm.

6.2. *The method*

We take the filter $\mathcal{F}$ to be a set of pairs $(g^{\mathcal{F}}, h^{\mathcal{F}}) \in \mathbb{R} \times [0, \infty)$. Then (g, h) is considered *acceptable to the filter* if for prescribed values of $\delta_{\mathcal{F}} \in (0, 1)$ and $\theta_{\mathcal{F}} > 0$, we have

$$\text{for all } (g^{\mathcal{F}}, h^{\mathcal{F}}) \in \mathcal{F} \quad \text{either} \quad g \leq g^{\mathcal{F}} - \theta_{\mathcal{F}} h^{\mathcal{F}} \quad \text{or} \quad h \leq (1 - \delta_{\mathcal{F}}) h^{\mathcal{F}}.$$

By *augmenting the filter* with (g, h) we mean replacing it with

$$\{(g, h)\} \cup \{(g^{\mathcal{F}}, h^{\mathcal{F}}) \in \mathcal{F} \mid g^{\mathcal{F}} < g \text{ or } h^{\mathcal{F}} < h\}.$$

The first part of the following lemma is standard:

Lemma 6.1: *Suppose points added to the filter satisfy $g \in [\underline{g}, \overline{g}] \subset (-\infty, \infty)$ and $h \geq \underline{h} > 0$. Then the filter may be augmented only finitely many times with acceptable points (g, h). If, furthermore, $h \leq \overline{h}$, then the filter may be augmented at most $[(\overline{g} - \underline{g})/(\theta_{\mathcal{F}} \underline{h}) + 1][\delta_{\mathcal{F}}^{-1} \log \tau^{-1} + 1]$ times for $\tau := \underline{h}/\overline{h}$. In particular, if $\overline{g} - \underline{g} = O(\overline{h})$, then we have the bound $O(\tau^{-1} \log \tau^{-1})$ for the number of augmentations.*

Proof: Consider the square $A := [\underline{h}, \overline{h}] \times [\underline{g}, \overline{g}]$. It is covered by the rectangles $(\overline{h}(1 - \delta_{\mathcal{F}})^n [1 - \delta_{\mathcal{F}}, 1]) \times (\overline{g} - \theta_{\mathcal{F}} \underline{h}[k, k + 1])$, where $n = 0, 1, \ldots, N - 1$, and $k = 0, 1, \ldots, K - 1$. At most one point acceptable to the filter may lie in each rectangle, so the number of rectangles KN gives an upper bound on the number of acceptable points that may be inserted in the filter. Solving $\underline{g} > \overline{g} - \theta_{\mathcal{F}} \underline{h} K$, we get $K > (\overline{g} - \underline{g})/(\theta_{\mathcal{F}} \underline{h})$. Solving for N from $\underline{h} > (1 - \delta_{\mathcal{F}})^N \overline{h}$, we get the *sufficient* condition $N > \log(\overline{h}/\underline{h}) \delta_{\mathcal{F}}^{-1}$ (by application of $-\log(1 - \delta_{\mathcal{F}}) \geq \delta_{\mathcal{F}}^2/2 + \delta_{\mathcal{F}} \geq \delta_{\mathcal{F}}$). This gives the desired bound in the case $h \leq \overline{h}$.

Suppose then that there's an infinite sequence $(h_{[k]}, g_{[k]}) \in \mathcal{F}$, $k = 1, 2, \ldots$, with $h_{[k+1]} \geq h_{[k]}$. Then $g_{[k+1]} \leq g_{[k]} - \theta_{\mathcal{F}} h_{[k]} \leq g_{[k]} - \theta_{\mathcal{F}} \underline{h}$, so that $g_{[k+1]} \leq g_{[1]} - k\theta_{\mathcal{F}} \underline{h}$, and for large enough k, $g_{[k+1]} < \underline{g}$, which is a contradiction. Therefore there exists some finite $\overline{h} \geq h$, and only finitely many entries may be added in the filter. □

In our present situation, we take $g = f_\nu(y)$ as the quality of the solution in terms of objective function value, and $h = \epsilon = r\mu$ as the quality of the solution in terms of $0 \in \partial_\epsilon^{\mathrm{DC}} f_\nu(y)$, as in Algorithm 4.1. Therefore, in contrast to the situation in constrained optimisation, either filter element becoming sufficiently small provides an approximate solution of prescribed quality. Unless the restoration method fails (which our restoration method of choice will not do), it always generates either a point acceptable to the filter, or a solution of such prescribed quality, by reducing the value of f_ν or ϵ sufficiently. Therefore, Lemma 6.1 alone proves convergence of the filter method in case of non-failure, if we augment the filter with acceptable points between restoration steps.

A crude filter method would therefore simply augment the filter and enter the restoration phase, whenever the main interior point method does not provide sufficient decrease in ϵ (or sufficiently long step), or the candidate iterate is unacceptable to the filter. The primary design goal of the filter method would, however, be to obtain greater practical convergence speeds than the pure restoration method or this crude filter method. We do not, however, concentrate on this paper on finding the best possible restoration and filter augmentation strategies. Rather, we concentrate on the theoretical aspects of using interior point SCP for restoration after presenting a method based on an approach familiar from other filter methods in the literature. The idea is to choose a shorter step size than allowed by the pure interior point method, if f_ν is sufficiently descending in the search direction. Also, if a linear model of the function does not predict decrease, we augment the filter for future reference.

In the rest of this section, we assume that both f and ν are of the form (3).

Suppose $y, \Delta y \in \mathbb{R}^m$ are given, and $0 \in \partial_\epsilon^{\mathrm{DC}} f(y)$. For arbitrary $z \in \partial^{\mathrm{DC}} f_\nu(y)$, we define the linear model of f_ν,

$$l(\alpha) := f_\nu(y) + \alpha\langle z, \Delta y\rangle.$$

We say that the model decreases sufficiently, if for prescribed $\kappa > 0$,

$$l(0) - l(\alpha) \geq \kappa\epsilon, \tag{31}$$

and that f_ν itself decreases sufficiently wrt. the model, if for given $\eta > 0$,

$$\frac{f_\nu(y) - f_\nu(y(\alpha))}{l(0) - l(\alpha)} \geq \eta. \tag{32}$$

Here we denote $y(\alpha) := y + \alpha\Delta y$ akin to (19). We also introduce the notation $\epsilon(\alpha) := \langle p(\alpha), d(\alpha)\rangle = r\mu(\alpha)$, where $\mu(\alpha)$ is given by (20).

With these definitions, the filter method is as follows.

Algorithm 6.1: Filter method for DC problems on symmetric cones

(1) Choose target accuracy $\underline{\epsilon} > 0$, parameters $\delta, \delta_{\mathcal{F}} \in (0,1)$, $\theta_{\mathcal{F}} > 0$, $\eta \in (0,1)$, and $\kappa > 0$, as well as the filter $\mathcal{F}$ and its initial contents.
(2) Initialise the interior point method per instructions of Algorithm 4.1 for the data of f_ν, yielding (p, d, y, ϵ) with $(p, d) \in G_\epsilon^{-1}(y, 0) \cap \mathcal{C}_\bullet(\gamma)$.
(3) If $\epsilon \leq \underline{\epsilon}$, stop, for we have a solution.
(4) Calculate the direction $(\Delta p, \Delta d, \Delta y)$ by solving, as in Algorithm 4.1, a scaled version of (16)–(18). Set $\alpha := \hat{\alpha}$ with the latter as in Lemma 4.2.
(5) If Step (4) failed, or $\epsilon(\alpha)/\epsilon > 1 - \delta$, augment $\mathcal{F}$ with $(f_\nu(y), \epsilon)$, and enter the restoration phase that either
    a) Produces a new iterate (p, d, y, ϵ) with $(p, d) \in G_\epsilon^{-1}(y, 0) \cap \mathcal{C}_\bullet(\gamma)$ and $(f_\nu(y), \epsilon)$ acceptable to the filter. In this case we continue from Step (4).
    b) Detects an $\underline{\epsilon}$-semi-critical point (or fails), in which case we stop.
(6) If $(f_\nu(y(\alpha)), \epsilon(\alpha))$ is acceptable to $\mathcal{F}$, and either (31) fails or (32) holds, go to Step (8).
(7) Set $\alpha := \alpha/2$, and go to Step (5).
(8) If (31) fails, augment $\mathcal{F}$ with $(f_\nu(y), \epsilon)$.
(9) Update $(p, d, y, \epsilon) := (p(\alpha), d(\alpha), y(\alpha), \epsilon(\alpha))$, and continue from Step (3).

Theorem 6.2: *Suppose the filter $\mathcal{F}$ is initialised to include $\{(\overline{g}, 0)\}$ for some $\overline{g} > \min f_\nu$ (and that the initial iterate is acceptable to $\mathcal{F}$). Then Algorithm 6.1 converges in a finite number of iterations to an $\underline{\epsilon}$-semi-critical point (if the restoration method does not fail). If, furthermore, always $\epsilon \leq \overline{\epsilon}$ for some $\overline{\epsilon} > \underline{\epsilon}$ such that $\overline{\epsilon} > \overline{g} - \min f_\nu$, and the restoration method is taken as an oracle, then the number* *of iterations is* $O(\tau^{-1}(\log\tau^{-1})^2)$ *for* $\tau := \underline{\epsilon}/\overline{\epsilon}$.

Proof: Step (5) ensures $\epsilon(\alpha)/\epsilon \le 1-\delta$. Thus a standard argument (cf. Lemma 6.1) shows that there are at most $O(\log\tau^{-1})$ iterations of the main phase of the algorithm between each restoration phase. Since the filter is augmented before each restoration phase with a point acceptable to it, Lemma 6.1 says that the restoration method may be called only a finite number of times. Furthermore, when $\epsilon \le \overline{\epsilon}$, Lemma 6.1 with $\underline{g} = \min f_\nu$ provides the bound $O(\tau^{-1}\log\tau^{-1})$ for the number of augmentations. □

Remark 6.1: Instead of directly specifying δ, we could specify $\beta \in (0,1)$, and calculate δ^{-1} according to Lemma 4.5. In this case we should include in the complexity estimate, the contribution by r, and potentially γ as well, depending on whether reinitialisation of $(p,d) \in G_\epsilon^{-1}(y,0) \cap \mathcal{C}_\bullet(\gamma)$ in the restoration method allows free choice, or guarantees a bound.

6.3. *Application of SCP to restoration phase*

A variant of Algorithm 5.1 can be used for restoration in Algorithm 6.1, and it never fails, so that convergence is attained. We simply add after Step (4) (of Algorithm 5.1) the step:

4^+. Calculate (p,d,ϵ) such that $(p,d) \in G_\epsilon^{-1}(\hat{y},0)\cap\mathcal{C}_\bullet(\gamma)$ (for the data of f_ν). If $(f_\nu(\hat{y}),\epsilon)$ is acceptable to $\mathcal{F}$, return to the main phase with result $(p,d,\hat{y},\epsilon)$.

If the basic version of Algorithm 5.1 is used (or the variant of Remark 5.1) then provided that $\overline{\epsilon}$ is large enough that the initialisation required by Theorem 5.1 can be performed (cf. Lemma 5.7), we have the bound $O(K_{\gamma_f,r_f}\tau_\rho^{-1}\log\tau_\rho^{-1})$ with $\tau_\rho := (\underline{\epsilon}-2\rho)/\overline{\epsilon}$ for the number of interior point iterations in each restoration phase. Since $\tau_\rho \le \tau = \underline{\epsilon}/\overline{\epsilon}$, the total number of interior point iterations in Algorithm 6.1 (with those in the main phase for f_ν, and those in the restoration phase for f alone), is therefore bounded by $O(K_{\gamma_f,r_f}\tau_\rho^{-2}(\log\tau_\rho^{-1})^3)$, provided that the conditions in Theorem 6.2 are satisfied, including $\epsilon \le \overline{\epsilon}$ on return from Step 4^+ above.

This bounded reinitialisation in Step 4^+ can indeed be enforced by adding such a check (or including $(0,\overline{\epsilon})$ in the filter), in which case the SCP restoration method simply churns out new candidates while decreasing f_ν, until it reaches an $\underline{\epsilon}$-semi-critical point or an acceptable candidate. The check does not degrade the complexity bounds calculated above, because SCP alone has lower complexity. The complexity of the method is thus entirely dependent on τ, the worst initialisation quality proportional to the desired solution quality, and ψ, which describes the proportion of the concave component and closeness to level-unboundedness of f_ν.

We may, however, also calculate some bounds for reinitialisation quality, to ensure that provided with big enough but reasonably bounded $\overline{\epsilon}$ and γ, the enforcement of $\epsilon \le \overline{\epsilon}$ does not simply reduce the filter method to SCP. The next result proves the existence of such a "good" initialiser; later a more practical procedure is provided, with bounds not so directly related to the quality of the current iterate. Note from the proof that the bounds are also good for initialisation (of f data) for SCP restoration, in addition to reinitialisation (of f_ν data) on return to the primary phase.

Theorem 6.3: *Fix the constants* $\underline{\epsilon} \ge 2\rho > 0$. *Suppose Assumption 5.4 holds for* f *and* $\mathcal{R}(\partial\nu) \subset \psi\mathcal{R}(\partial f)$ *for some* $\psi \in (0,1)$. *Suppose moreover that* $f_\nu(y) - \min f_\nu \le \Delta_0$. *Then either of the following holds:*

(i) y *is* $\underline{\epsilon}$*-semi-critical for* f_ν.

(ii) *There exists* $(p, d) \in G_{\bar\epsilon}^{-1}(y, 0) \cap \mathcal{C}_{-\infty}(\gamma)$ *for* $\bar\epsilon = O(\Delta_0 + \|V_f\|_F\|c_f\|_F)$, $\gamma \in [0, 1)$ *with* $(1-\gamma)^{-1} = O((1-\psi)^{-2}\tau^{-1})$, *and* $\tau := \underline{\epsilon}/\bar\epsilon$.

Proof: Find $z \in \mathbb{R}^m$ and $(p_\nu, d_\nu) \in G_{\nu,\rho}^{-1}(y, z) \cap \mathcal{C}_{-\infty}(\psi)$ with *exactly* $\langle p_\nu, d_\nu\rangle = \rho$. This can be done, even with $\psi = 0$, because the selection $p_\nu \circ d_\nu = \mu_\nu e$, with $\mu_\nu := \rho/r_\nu$, within $\partial\nu$ comes from the subdifferential of a barrier-smoothed function; cf. Remark 3.1. An alternative way to see this, is to write $\xi_\nu := -B_\nu^* y - c_\nu$, to get the system of equations

$$A_\nu^*\lambda_\nu + d_\nu = \xi_\nu',\ A_\nu p_\nu = b_\nu,\ p_\nu \circ d_\nu = \mu_\nu e;\ p_\nu, d_\nu \in \mathcal{K}_\nu, \tag{33}$$

which characterises the solutions of (cf. e.g. [9, 26])

$$\min\ [\langle \xi_\nu, p_\nu\rangle - \mu_\nu \log(\det p_\nu)] \quad \text{subject to} \quad A_\nu p_\nu = b_\nu,\ p_\nu \in \mathcal{K}_\nu.$$

With z and (p_ν, d_ν) found, apply Lemma 5.7 to find $(p_f, d_f) \in G_{f,\epsilon}^{-1}(y, z) \cap \mathcal{C}_{-\infty}(\gamma')$ for some $\epsilon = \langle p_f, d_f\rangle = O(\Delta_0 + \rho + \|V_f\|_F\|c_f\|_F)$, and $\gamma' \in [0, 1)$ with $(1-\gamma')^{-1} = O((1-\psi)^{-1})$. Apply the following Lemma 6.4, to get the claim of the theorem at y for $\epsilon = O(\bar\epsilon) := O(\Delta_0 + 2\rho + \|V_f\|_F\|c_f\|_F)$ and $\tau_y^{-1} = O(\tau^{-1})$. Finish the proof by referring to Remark 5.3 to take out ρ from the complexity. □

Lemma 6.4: *Assume we have fixed* $\underline{\epsilon} \geq 2\rho \geq \theta\underline{\epsilon} > 0$ *for some* $\theta > 0$. *Suppose that for some* $\gamma' \in [0, 1)$ *and* $\epsilon' > 0$, *we have* $(p_f, d_f) = G_{f,\epsilon'}^{-1}(y, z) \cap \mathcal{C}_{-\infty}(\gamma')$ *and* $(p_\nu, d_\nu) = G_{f,\rho}^{-1}(y, z) \cap \mathcal{C}_{-\infty}(\gamma')$ *with* exactly $\epsilon' = \langle p_f, d_f\rangle$ *and* $\rho = \langle p_\nu, d_\nu\rangle$. *Then either of the following holds:*

(i) $\epsilon' + \rho \leq \underline{\epsilon}$, *in which case* y *is* $\underline{\epsilon}$*-semi-critical for* f_ν.

(ii) $(p, d) = ((p_f, p_\nu), (d_f, d_\nu)) \in G_\epsilon^{-1}(y, 0) \cap \mathcal{C}_{-\infty}(\gamma)$ *for* $\epsilon := \epsilon' + \rho$, *and* $\gamma \in [0, 1)$ *with* $(1-\gamma)^{-1} = O((1-\gamma')^{-2}\tau_y^{-1})$, *and* $\tau_y := \underline{\epsilon}/\epsilon$.

Proof: Let $q = (q_f, q_\nu) := (Q_{p_f}^{1/2} d_f, Q_{p_\nu}^{1/2} d_\nu) = Q_p^{1/2} d$. Denoting $\underline{\zeta}(q) := \min_i \zeta_i(q)$, we have

$$(1-\gamma')\mu(q_f) \leq \underline{\zeta}(q_f) \leq \mu(q_f), \tag{34}$$

and likewise for ν. Therefore,

$$(1-\gamma')\mu(q) = (1-\gamma')\frac{r_f\mu(q_f) + r_\nu\mu(q_\nu)}{r_f + r_\nu} \leq \frac{r_f\underline{\zeta}(q_f) + r_\nu\underline{\zeta}(q_\nu)}{r_f + r_\nu}.$$

But employing (34) and the exactness assumption, we have

$$r_\nu\underline{\zeta}(q_\nu)/\underline{\zeta}(q_f) \leq r_\nu\frac{\mu(q_\nu)}{\mu(q_f)}/(1-\gamma') \leq r_f\frac{\rho}{\epsilon'}/(1-\gamma'),$$

Because an analogous estimate holds with the roles of f and ν reversed, and $\underline{\zeta}(q) = \min\{\underline{\zeta}(q_f), \underline{\zeta}(q_\nu)\}$, we get

$$(1-\gamma')\mu(q) \leq \frac{1 + \max\{\epsilon'/\rho, \rho/\epsilon'\}}{1-\gamma'}\underline{\zeta}(q).$$

If $\epsilon' \leq \rho$, then $\epsilon' + \rho \leq 2\rho \leq \underline{\epsilon}$, which is covered by case (i). So assume the contrary.

We now get $1+\epsilon'/\rho = (\rho+\epsilon')/\rho \le \epsilon/(\theta\underline{\epsilon})$. Therefore, with γ defined by $(1-\gamma)^{-1} = (1+\epsilon'/\rho)(1-\gamma')^{-2}$, we have $(1-\gamma)^{-1} = O((1-\gamma')^{-2}\tau_y^{-1})$, as well as $\gamma \in [0,1)$ and $(1-\gamma)\mu(q) \le \underline{\zeta}(q)$. Hence, case (ii) applies. □

We can in principle solve (33) approximately by standard interior point methods. After all, instead of $p_\nu \circ d_\nu = \mu_\nu e \in \mathcal{C}_{-\infty}(0)$, we only wanted $\mathcal{C}_{-\infty}(\psi)$. Then we could calculate (p_f, d_f) and modify the result as indicated in the proof. However, we would have to bound the quality of the initialisation for this method, which would annoyingly seem to involve y or the linearisation error ℓ (to be defined below). Sometimes (33) can be solved directly, however, as the examples below show. After that, we would still have to find $(p_f, d_f) \in G^{-1}_{f,\epsilon'}(y,z) \cap \mathcal{C}_{-\infty}(\gamma')$ as discussed towards the end of Section 5.2 above.

Example 6.5 Suppose Assumption 5.4 holds for ν. Let $\xi_\nu = -B^*_\nu y - c_\nu$ be as in Theorem 6.3. Then, dropping the ν-subscripts to simplify the notation for this example, $d_i = \xi_i + \lambda_i a'_i$ and $p_i = \mu d_i^{-1}$, assuming λ_i is big enough for d_i to be invertible. The problem now is to have $\langle a'_i, p_i\rangle = b_i$, i.e. $\operatorname{tr}(Q^{1/2}_{a'_i} d_i^{-1}) = b_i/\mu$. Taking $Q^{1/2}_{a'_i}$ inside d_i^{-1} (which can be done by e.g. [7, Proposition II.3.3]), we get

$$\operatorname{tr}(Q^{-1/2}_{a'_i}\xi_i + \lambda_i e)^{-1} = b_i/\mu. \tag{35}$$

Thus we can solve (33) if we can invert the trace of the *resolvents* of $\underset{\sim}{\xi}_i := Q^{-1/2}_{a'_i}\xi_i$.

Example 6.6 Suppose that (each) $\mathcal{K}_i$ in Example 6.5 is a second order cone. Then for $x = (x^0, \bar{x})$, we have $x^{-1} = (x^0, -\bar{x})/\det(x)$, $\det(x) = (x^0)^2 - \|\bar{x}\|^2$, and $\operatorname{tr} x = \langle e, x\rangle = 2x^0$. By Assumption 5.4, $\langle e, \underset{\sim}{\xi}_i\rangle = \langle (a'_i)^{-1}, \xi_i\rangle = 0$, which implies $\underset{\sim}{\xi}{}_i^0 = 0$. Therefore, $\operatorname{tr}(\underset{\sim}{\xi}_i + \lambda_i e)^{-1} = 2\lambda_i/(\lambda_i^2 - \|\bar{\underset{\sim}{\xi}}_i\|^2)$, so we get from (35) the quadratic equation $(2\mu_\nu/b_i)\lambda_i = \lambda_i^2 - \|\bar{\underset{\sim}{\xi}}_i\|^2$. This can be solved for λ_i, as we wanted.

The proof of the next result provides a simpler reinitialisation method, although with worse bounds near an actual minimum of f_ν. The practical performance sometimes seems better, however. The subgradient assumptions are guaranteed by the SCP procedure.

Lemma 6.7: *Suppose Assumption 5.4 holds (for both f and ν), and that $z \in \partial f_{\epsilon'}(\hat{y})$ and $z \in \partial_\rho \nu(y)$. Denote the linearisation error of ν by $\ell := \nu(\hat{y}) - \nu(y) - z^T(\hat{y} - y)$. Then for all $\psi \in (0,1)$, there exist $(p,d) \in G^{-1}_\epsilon(\hat{y}, 0) \cap \mathcal{C}_{-\infty}(\gamma)$ with $\gamma := (1+\psi)/2$ and $\epsilon/2 := \psi(\epsilon' + \rho + \ell) + (1-\psi)(f(\hat{y}) + \nu(\hat{y}))$.*

Proof: We note that by the definition of f, there exists $\hat{p}_f(y) \in V_f$ such that $f(y) = \langle B^*_f y + c_f, \hat{p}_f(y)\rangle$. Furthermore, by (7), there exists $\hat{d}_f(y) = -B^*_f y - c_f - A^*_f \hat{\lambda}_f(y) \in \mathcal{K}_f$ such that $\langle \hat{p}_f(y), \hat{d}_f(y)\rangle = 0$. Therefore, for all $p'_f \in V_f$

$$f(y) - \langle B^*_f y + c_f, p'_f\rangle = \langle \hat{d}_f(y), \hat{p}_f(y) - p'_f\rangle = \langle \hat{d}_f(y), p'_f\rangle. \tag{36}$$

An analogous result holds for ν.

By the approximate subgradient transportation formula [15, Proposition XI.4.2.4], $z \in \partial\nu_{\rho+\ell}(\hat{y})$. Therefore, we can find $(p', d') \in G^{-1}_{\epsilon'+\rho+\ell}(\hat{y}, 0)$. In fact, we can take $d' = \hat{d} := (\hat{d}_f(\hat{y}), \hat{d}_\nu(\hat{y}))$, since with $\hat{y}$ fixed, the choice $\hat{\lambda}(\hat{y})$ for λ must minimise $d' \mapsto \langle d', p'\rangle$. (If some other d' at y achieved lower value, then also $\langle d', \hat{p}\rangle < \langle \hat{d}, \hat{p}\rangle = 0$, which is a contradiction to properties of symmetric cones.) We therefore have by Assumption 5.4 and (36) with $p' = a^{-1}$ that

$$\langle a^{-1}, d' \rangle = f(\hat{y}) + \nu(\hat{y}) - \langle B^* \hat{y} + c, a^{-1} \rangle = f(\hat{y}) + \nu(\hat{y}). \tag{37}$$

Applying Lemma 5.6 with $\gamma' = 1$ and $\gamma = (1+\psi)/2$ now yields the claim for

$$\epsilon/2 = \psi \langle p', d' \rangle + (1-\psi)\langle a^{-1}, d' \rangle.$$

It only remains to use (37) and $\langle p', d' \rangle \leq \epsilon' + \rho + \ell$. □

Remark 6.2: The subgradient transportation formula actually holds for fixed p'_ν. To see this, suppose $(p'_\nu, d'_\nu) \in G^{-1}_{\nu,\rho}(y,z)$ and calculate $\langle p'_\nu, \hat{d}_\nu(\hat{y}) \rangle = \langle p'_\nu, \hat{d}_\nu(y) \rangle + \langle p'_\nu, \hat{d}_\nu(\hat{y}) - \hat{d}_\nu(y) \rangle = \langle p'_\nu, \hat{d}_\nu(y) \rangle + \nu(\hat{y}) - \nu(y) - \langle B^*_\nu(\hat{y} - y), p'_\nu \rangle \leq \rho + \ell$, where we have applied (36) twice in the last equality.

This means that we can with simple modifications of (p'_ν, d'_ν) and $(p'_f, d'_f) \in G^{-1}_{f,\epsilon'}(\hat{y}, z)$, produce (p, d) satisfying the claims of Lemma 6.7: calculate $\hat{d}(\hat{y})$, translate $p' = (p'_f, p'_\nu)$ towards a^{-1} by $1-\psi$, and add a factor of a to $\hat{d}(\hat{y})$.

Remark 6.3: As we see, to ensure that $(p, d) \in \mathcal{C}_{-\infty}(\gamma)$, without any further knowledge of the containment in $\mathcal{C}_\bullet(\gamma')$ of $(p'_\nu, d'_\nu) \in G^{-1}_{\nu,\rho}(\hat{y}, z)$ after transportation of z from y to $\hat{y}$, we have to ensure that p is also far enough from the boundary of $\mathcal{K}$. To do so, we apply the translation towards a^{-1}. But this component brings the annoying $f + \nu$ sum (instead of difference) into the bound, which is not found in the bound of Theorem 6.3.

7. Preliminary practical experience

As the performance of the algorithms does not appear stellar at this stage of development, we have chosen to leave statistically significant testing outside the scope of this mainly theoretical paper. In this section, we however list some observations from our limited experience with the methods, helpful for further work.

The primary applications we had in mind in the study of Algorithms 4.1 and 6.1 were the K-spatial-median or multisource Weber problem, as well as the MO clustering formulation from [30], along with reformulations of the Euclidean TSP based on these clustering objectives from [29]. The MO clustering problem reads with the notation $\bar{y} = (y_1, \ldots, y_s) \in \mathbb{R}^{sm}$, $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{R}^{nm}$ as

$$\min_{\bar{y}} f(\bar{y}; \bar{a}) - w\nu(\bar{y}) \tag{38}$$

for some $w \in (0, n/(s-1))$, and

$$f(\bar{y}; \bar{a}) := \sum_{i=1}^{s} \sum_{k=1}^{n} \|y_i - a_k\|, \quad \nu(\bar{y}) := \sum_{i<j} \|y_i - y_j\|.$$

In the MO-TSP problem, we set $s = n$, $w = 1$, and add to (38), the path-length penalty $\lambda_{\mathrm{TSP}} f_{\mathrm{TSP}}(\bar{y})$ for some $\lambda_{\mathrm{TSP}} > 0$ and $f_{\mathrm{TSP}}(\bar{y}) := \sum_{i=1}^{n} \|y_i - y_{i+1}\|$ (with the identification $y_{n+1} = y_1$).

According to [30], $n(s-1)^{-1}\mathcal{R}(\partial\nu) \subset \mathcal{R}(\partial f)$. Therefore, by our choice of $w = 1$ in the TSP problem, we may take $\psi = (n-1)/n$ in Lemma 5.7 and Theorem 6.3, and obtain $(1-\gamma)^{-1} = O((1-\psi)^{-1}) = O(n)$. Thus the complexity of the method in this application only depends polynomially on n (through both $r =$

$2(n^2 + n(n-1)/2 + n) = 3n^2 + n$ and γ), and log-polynomially on the reciprocal of the desired relative solution quality τ. Recall from Example 5.8 that as a sum of spatial medians, finding $p_f \in V_f$ satisfying $B_f p_f = z$ is easy, while we may choose $B_{\text{TSP}} p_{\text{TSP}} = 0$ ($p_{\text{TSP}} = e$) for the f_{TSP} component.

Our principal observations from applying Algorithm 6.1 to these problems are:

(i) Good reinitialisation after restoration is difficult, resulting in unpredictable performance. Often the filter method never succeeds in an interior point step after returning from a restoration phase, and therefore performs worse than plain SCP (which offers quite consistent but not stellar performance). At other times the method performs well as wanted: some initial runs of the restoration method find a good spot for local convergence of the main interior point method, and the final steps are fast.

(ii) The spatial median of the data $\bar{a}$ is quite attractive: Unless care is taken to (re)initialise the method with p and d (and not just y) close to some other attractor (semi-critical point, cf. Lemma 3.8), it is likely that some of the variables y_i will converge to the spatial median. Especially this appears to be a problem when s is a considerable proportion of n, such as in the MO-TSP case. The filter can of course be initialised to forbid such convergence, but this may provoke long restoration runs.

In summary, we find that although the theoretical basis of our method is sound, more research and experimentation is still needed to find out if and with what parametrisation and modifications, the filter algorithm can provide competitive practical performance in these, and other, applications.

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