Sparse rectifiability and compactness in $\mathrm{SBV}(\Omega)$

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Abstract

We introduce a notion of sparse rectifiability, stronger than that of uniform rectifiability. As applications we derive, firstly, results ensuring the convergence of the total variation measures $|\mu^i|$ subject to the weak* convergence of the sparsely rectifiable Radon measures μ^i. Secondly, we apply sparse rectifiability to derive compactness results for special functions of bounded variation (SBV) and, more generally, special functions of bounded deformation (SBD).

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1. Introduction

Suppose μ is a Radon measure on $\mathbb{R}^m$. As a simplified variant of the definition, we then say that μ is *sparsely rectifiable* (in dimension d), if it is upper Ahlfors-regular and there exists a family $\mathcal{G} = \{\Gamma_j^x \mid x \in \mathbb{R}^m,\ j = 0, 1, 2, \ldots\}$ of Lipschitz d-graphs of constant at most L, satisfying

$$\mathrm{Sp}(\mu; \mathcal{G}) := \sum_{j=0}^{\infty} 2^{jm} \int |\mu|([0, 2^{-j}]^m \setminus \Gamma_j^x)\, dx < \infty.$$

Roughly speaking, local deviations of μ from a Lipschitz graph, in the sense of measure, vanish as the scale gets smaller.

It turns out, as we show in this paper, that sparsely rectifiable measures with compact support are rectifiable. Indeed, subject to lower Ahlfors-regularity, sets E such that $\mathcal{H}^d \llcorner E$ is sparsely rectifiable, are also uniformly rectifiable per [7]. Moreover, weak* limits of sequences of suitably bounded sparsely rectifiable Radon measures remain sparsely rectifiable.

Besides these basic results on sparsely rectifiable measures, our intent is to apply this new notion to derive compactness and approximability results for sequences of special functions of bounded variation (SBV, see [2]). Analogous results hold for special functions of bounded deformation (SBD, see [9, 1]); we however concentrate on the former due to greater familiarity for most readers. Specifically, by bounding

$$\sup_i \|u^i\|_{L^1(\Omega)} + \int_\Omega \psi(\nabla u^i(x))\, dx + |D^j u^i|(\Omega) + \mathrm{Sp}(D^j u^i; \mathcal{G}^i) < \infty,$$

we are able to derive a basic compactness result in $\mathrm{SBV}(\Omega)$ with convergence properties comparable to the SBV compactness theorem of Ambrosio (see [2]) or the analogous result of Bellettini et al. [3]

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for SBD. Indeed, the only difference is that instead of bounding and showing lower-semicontinuity of the mass $\mathcal{H}^{m-1}(J_{u^i})$ of the jump sets, we do the same for $\mathrm{Sp}(D^j u^i; \mathcal{G}^i)$.

The convergence of the jump parts $D^j u^i$ of the distributional gradients to $D^j u$ in the basic compactness result is only weak*. We are however able to get improved forms of convergence by employing the following regularisation result. Specifically, let $\{\mu^i\}_{i=0}^{\infty}$ be a sequence of Radon measures, convergent weakly* to some Radon measure μ. We introduced in [10] a regularisation term $\eta(\mu^i)$, whose boundedness ensures the weak* convergence of $|\mu^i|$ to $|\mu|$. Applying these results in conjunction with (either of) the SBV compactness results discussed above, we can then formulate, for example, a compactness result with the jump divergence $\mathrm{Div}^j u^i := \mathrm{Tr}\, D^j u^i$ convergent in total variation. Indeed, the motivation for the present study stems from an optimisation problem studied in [10], where we need this kind of stronger convergence from $\mathrm{Div}^j u^i$, with $u^i \in \mathrm{SBD}(\Omega)$. This is to ensure the closedness of a PDE constraint (an extended form of the transport equation with "jump sources and sinks"), needed to show the existence of solutions to the problem.

Additionally in this paper, based on the sparse rectifiability of μ, we derive more easily calculable bounds for $\eta(\mu)$. A still unanswered question is whether we can construct a sequence of (discrete) approximations $\{u^i\}_{i=0}^{\infty}$ of u such that $F(u^i) \to F(u)$ for functionals F involving terms like $\eta(\mathrm{Div}^j u)$. By using sparse rectifiability estimates, we expect to be able to carry this out. In this paper we study some conditions ensuring that $\eta(\mathrm{Div}^j u^i)$ is convergent for a given approximating sequence. To keep the paper focussed and at a reasonable length, we however leave the construction of a concrete approximating sequence to future studies.

The rest of this paper is organised as follows. In Section 2 we introduce notation, spaces and other tools used. Then in Section 3 we study sparse rectifiability. Section 4 is an interlude that introduces some Poincaré inequalities used in the following Section 5 that where we study the η functional. Finally, in Section 6 we study compactness in SBV/SBD.

2. Notation and other preliminaries

2.1. Sets and functions

We denote the open ball of radius ρ centred at $x \in \mathbb{R}^m$ by $B(x,\rho)$. For $\nu \in \mathbb{R}^m$, we denote the orthogonal hyperplane by $\nu^\perp := \{z \in \mathbb{R}^m \mid \langle \nu, z\rangle = 0\}$. P_ν denotes the projection onto the subspace spanned by ν, and by $P_\nu^\perp$ the projection onto the orthogonal subspace. Given a set $A \subset \mathbb{R}^m$, we denote by $\mathrm{conv}\, A$ the convex hull of A.

The trace of a matrix $A \in \mathbb{R}^{m\times m}$ is denoted $\mathrm{Tr}\, A$, and the k-dimensional Jacobian of a linear map $L : \mathbb{R}^k \to \mathbb{R}^m$, $(k \le m)$, is defined as $\mathcal{J}_k[L] := \sqrt{\det(L^* \circ L)}$.

A set $\Gamma \subset \mathbb{R}^m$ is a called a Lipschitz d-graph (of Lipschitz factor L), if there exist a unit vector z_Γ, an open set V_Γ on a d-dimensional subspace of $(z_\Gamma)^\perp$, and a Lipschitz map $g_\Gamma : V_\Gamma \to \mathbb{R}^m$ of Lipschitz factor at most L, such that

$$\Gamma = \{y \in \mathbb{R}^m \mid g_\Gamma(v) = y,\, v = P_{z_\Gamma}^\perp y \in V_\Gamma\}.$$

Remark 1. If Γ is the graph of $f : U \subset \mathbb{R}^{m-1} \to \mathbb{R}$, then $g_\gamma(x,0) = (x, f(x))$ for $(x,0) \in V_\Gamma = U \times \{0\}$.

2.2. Measures

The space of (signed) Radon measures on an open set Ω is denoted $\mathcal{M}(\Omega)$. If V is a vector space, then the space of V-valued Radon measures on Ω is denoted $\mathcal{M}(\Omega; V)$. The k-dimensional Hausdorff measure, on any given ambient space $\mathbb{R}^m$, $(k \le m)$, is denoted by $\mathcal{H}^k$, while $\mathcal{L}^m$ denotes the Lebesgue measure on $\mathbb{R}^m$. For a measure μ and a measurable set A, we denote by $\mu \llcorner A$ the restriction measure defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$. The total variation measure of μ is denoted $|\mu|$.

A measure $\mu \in \mathcal{M}(\Omega)$ is said to be *Ahlfors-regular* (in dimension d), if there exists $M \in (0, \infty)$ such that
$$M^{-1} r^d \le |\mu|(B(x,r)) \le M r^d \quad \text{for all} \quad r > 0 \text{ and } x \in \operatorname{supp} \mu.$$
If only the first or second inequality holds, then μ is said to be, respectively, *lower* or *upper* Ahlfors-regular.

2.3. Functions of bounded variation

Following Ambrosio et al. [2], a function $u : \Omega \to \mathbb{R}^k$ on a bounded open set $\Omega \subset \mathbb{R}^m$, is said to be of *bounded variation*, denoted $u \in [\mathrm{BV}(\Omega)]^k$, if $u \in L^1(\Omega)$, and the distributional gradient Du is a Radon measure.

Given a sequence $\{u^i\}_{i=1}^\infty \subset [\mathrm{BV}(\Omega)]^k$, strong convergence to $u \in [\mathrm{BV}(\Omega)]^k$ is defined as strong L^1 convergence $\|u^i - u\|_{L^1(\Omega;\mathbb{R}^k)} \to 0$ together with convergence of the total variation $|u - u^i|(\Omega) \to 0$. Weak convergence is defined as $u^i \to u$ strongly in $L^1(\Omega; \mathbb{R}^k)$ along with $Du^i \overset{*}{\rightharpoonup} Du$ weakly* in $\mathcal{M}(\Omega; \mathbb{R}^{k \times m})$.

The distributional gradient be decomposed as $Du = \nabla u \mathcal{L}^m + D^j u + D^c u$, where the density ∇u of the *absolutely continuous part* of Du equals (a.e.) the approximate differential of u. The *jump part* $D^j u$ may be represented as
$$D^j u = (u^+ - u^-) \otimes \nu_{J_u} \mathcal{H}^{m-1} \llcorner J_u, \tag{1}$$
where x is in the *jump set* J_u of u if for some $\nu := \nu_{J_u}(x)$ there exist two distinct one-sided traces $u^\pm(x)$ defined as satisfying
$$\lim_{\rho \searrow 0} \frac{1}{\rho^m} \int_{B^\pm(x,\rho,\nu)} \|u^\pm(x) - u(y)\| \, dy = 0, \tag{2}$$
where $B^\pm(x, \rho, \nu) := \{y \in B(x, \rho) \mid \pm\langle y - x, \nu\rangle \ge 0\}$. It turns out that J_u is countably $\mathcal{H}^{m-1}$-rectifiable, and ν is (a.e.) the normal to J_u. The remaining *Cantor part* $D^c u$ vanishes on any Borel set σ-finite with respect to $\mathcal{H}^{m-1}$.

The space $[\mathrm{SBV}(\Omega)]^k$ of *special functions of bounded variation* is defined as those $u \in [\mathrm{BV}(\Omega)]^k$ with $D^c u = 0$. There we have the following compactness result.

Theorem 1 (SBV compactness [2])**.** *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$. If $\{u^i\}_{i=0}^\infty \subset [\mathrm{SBV}(\Omega)]^k$ with*
$$\|u^i\|_{L^1} + \int_\Omega \psi(|\nabla u^i|) \, dx + |D^j u^i|(\Omega) + \mathcal{H}^{m-1}(J_{u^i}) \le K < \infty.$$
Then there exists a subsequence of $\{u^i\}_{i=0}^\infty$, unrelabelled, such that
$$\begin{aligned} &u^i \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^k), \\ &\nabla u^i \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega; \mathbb{R}^{k\times m}), \\ &D^j u^i \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(\Omega; \mathbb{R}^{k\times m}), \text{ and} \\ &\mathcal{H}^{m-1}(J_u) \le \liminf_{i\to\infty} \mathcal{H}^{m-1}(J_{u^i}). \end{aligned}$$

3. Sparse rectifiability

We now introduce a quantifiable notion of "sparse" rectifiability, which bears some resemblance to definitions of uniform rectifiability, as studied by David and Semmes [7]. That notion, however, does not provide the regularity we need, as it allows considerable "dense" packing of the set, merely measuring locally the deviation from a Lipschitz surface in a geometric sense. Our notion of "sparse" rectifiability, by contrast, measures the deviation in the sense of measure.

Definition 1. A sequence $\{(f_j,\nu_j)\}_{j=0}^{\infty}$ of bounded Borel functions $f^j:\mathbb{R}^m\to\mathbb{R}$ and Borel probability measures ν^j on $\mathbb{R}^m$ is said to form a *nested sequence of functions* if $f_j(x)=\int f_{j+1}(x-y)\,d\nu_j(y)$ (a.e.). The sequence is said to be *normalised* if $f_j\geq 0$ and $\int f_j\,dx=1$. The sequence is said to be *regular*, if it is normalised, and there exist constants $\alpha>0$ and $\beta>0$, and a sequence $h_j\searrow 0$, such that $\alpha h_j^{-m}\chi_{B(0,\beta h_j)}\leq f_j\leq \alpha^{-1}h_j^{-m}\chi_{B(0,h_j)}$.

Whenever $\{f_j\}_{j=0}^{\infty}$ is a nested sequence of functions, we denote $S_j^x:=x+\operatorname{supp} f_j$. We also employ the notation $\tau_x f(y):=f(y-x)$.

Remark 2. Regularity holds in the typical case $f_j(x):=h_j^{-m}f(x/h_j)$ for $h_j\searrow 0$ and some $f\geq \alpha\chi_{B(0,\beta)}$ with compact support and $\int f_j\,dx=1$. Examples include $f=\chi_{[-1/2,1/2]^m}$ in $\mathbb{R}^m$, and $f(t)=\max\{0,\min\{1+t,1-t\}\}$ in $\mathbb{R}$ (as well as similar but more complicated shape functions in $\mathbb{R}^m$).

Definition 2. Let $\Omega\subset\mathbb{R}^m$ open and bounded, and $\{f_j\}_{j=0}^{\infty}$ a *regular* nested sequence of functions per Definition 1. A Radon measure $\mu\in\mathcal{M}(\Omega)$ is said to be *sparsely rectifiable* in dimension $d\leq m-1$ with constants $L,M\in[0,\infty)$ (with respect to $\{f_j\}_{j=0}^{\infty}$), denoted $\mu\in\operatorname{Sp}^d(\Omega,L,M)$, if the following hold.

1. μ is upper Ahlfors-regular in dimension d with constant M.
2. There exist families $\mathcal{G}=\{\mathcal{G}_j\}_{j=0}^{\infty}$, $\mathcal{G}_j=\{\Gamma_j^x\mid x\in\mathbb{R}^m\}$ of d-dimensional Lipschitz graphs Γ_j^x, of Lipschitz factor at most L, satisfying

$$\operatorname{Sp}(\mu;\mathcal{G}):=\sum_{j=0}^{\infty}\operatorname{Sp}_j(\mu;\mathcal{G}_j)<\infty,\quad\text{where}\quad \operatorname{Sp}_j(\mu;\mathcal{G}_j):=\int_{\mathbb{R}^m}\left|\mu\llcorner S_j^x\setminus\Gamma_j^x\right|(\tau_x f_j)\,dx. \tag{3}$$

Definition 3. We also set

$$\operatorname{Sp}(\mu):=\inf_{\mathcal{G}}\operatorname{Sp}(\mu;\mathcal{G}),\quad\text{and}\quad \operatorname{Sp}_j(\mu):=\inf_{\mathcal{G}_j}\operatorname{Sp}(\mu;\mathcal{G}_j),$$

where the infimum is taken over all families of the type specified above.

Definition 4. A bounded set $E\subset\mathbb{R}^m$ is said to sparsely rectifiable in dimension d, if $\mathcal{H}^d\llcorner E$ is.

Definition 5. For the Lipschitz graphs Γ_j^x from Definition 2, we use the shorthand notations $V_j^x:=V_{\Gamma_j^x}$, $g_j^x:=g_{\Gamma_j^x}$, and $z_j^x:=z_{\Gamma_j^x}$ for the entities from the definition in Section 2.

Remark 3. Much of our results would hold for bi-Lipschitz images Γ_j^x, as we will mostly be relying on the maps g_j^x satisfying bi-Lipschitz properties. For simplicity we restrict ourselves to Lipschitz graphs, however.

Proposition 1. *Suppose $\Omega\subset\mathbb{R}^m$ is open and bounded, and $\mu\in\mathcal{M}(\Omega)$ satisfies* (3)*. Then μ is concentrated on a countably d-rectifiable set J. If μ is sparsely rectifiable (i.e., is also upper Ahlfors-regular in dimension $d\leq m-1$), then μ is d-rectifiable, $\mu\ll\mathcal{H}^d\llcorner J$.*

Proof. Let $\mathcal{G}$ be as in Definition 2. Let K be a compact set containing $\operatorname{supp}\mu+B(0,h_0)$. To construct rectifiable approximations of $\operatorname{supp}\mu$, we need a partially discrete approximation of the Lebesgue integral over K. Denoting by α and β the regularity constants for $\{f_j\}_{j=0}^{\infty}$ from Definition 1, we set $A_j:=B(0,\beta h_j)$. With j fixed for the moment, we then apply the Besicovitch covering theorem on the family $\{x + A_j \mid x \in K\}$ to obtain an at most countable (actually finite) set G_j, such that for a dimensional constant c_m, we have

$$\chi_K \le \sum_{\xi \in G_j} \tau_\xi \chi_{A_j} \le c_m.$$

It follows that

$$\mathcal{L}^m \ge c_m^{-1} \sum_{\xi \in G_j} \mathcal{L}^m \llcorner (\xi + A_j). \tag{4}$$

Moreover, from the regularity condition for f_j, there exists a constant $C_1 > 0$ dependent on α, β, and m alone, such that

$$\sum_{\xi \in G_j} \tau_\xi f_j \ge \sum_{\xi \in G_j} h_j^{-m} \alpha \tau_\xi \chi_{A_j} \ge h_j^{-m} \alpha \chi_K \ge C_1 / \mathcal{L}^m(A_j) \chi_K. \tag{5}$$

Now, with this preliminary setup taken care of, let us for any given $y \in A_j$ set $J_j^y := \bigcup_{x \in G_j + y} \Gamma_j^x$. Then J_j^y is $\mathcal{H}^d$-rectifiable and we may, using (4) and (5), approximate

$$\begin{aligned}
\mathrm{Sp}_j(\mu; \mathcal{G}_j) &= \int \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right| (\tau_x f_j)\, dx \\
&\ge c_m^{-1} \int_{A_j} \sum_{x \in y + G_j} \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right| (\tau_x f_j)\, dy \\
&\ge c_m^{-1} \int_{A_j} \sum_{x \in y + G_j} \left|\mu \llcorner \Omega \setminus J_j^y\right| (\tau_x f_j)\, dy \\
&\ge \frac{C_1}{c_m \mathcal{L}^m(A_j)} \int_{A_j} \left|\mu \llcorner \Omega \setminus J_j^y\right| (\tau_y \chi_K)\, dy \\
&\ge \frac{C_1}{c_m \mathcal{L}^m(A_j)} \int_{A_j} |\mu|(\Omega \setminus J_j^y)\, dy.
\end{aligned}$$

We thus deduce that there is a choice of $y_j \in A_j$ with

$$\mathrm{Sp}_j(\mu; \mathcal{G}_j) c_m C^{-1} \ge |\mu|(\Omega \setminus J_j^{y_j}).$$

Setting $J := \bigcup_{j=0}^\infty J_j^{y_j}$, it follows from observing

$$|\mu|(\Omega \setminus J_j^{y_j}) \ge |\mu|(\Omega \setminus J)$$

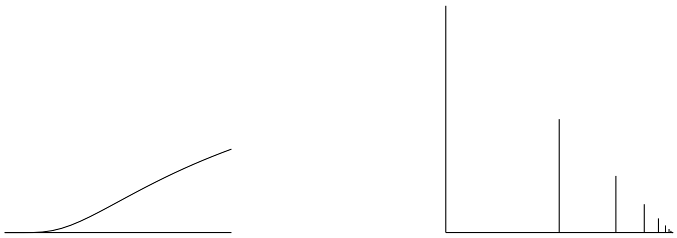
and letting $j \nearrow \infty$ that $|\mu|(\Omega \setminus J) = 0$. Since J is $\mathcal{H}^d$-rectifiable, this gives the first claim of the proposition. If $|\mu|$ is upper Ahlfors-regular in dimension d, we then have $|\mu| \ll \mathcal{H}^d \llcorner J$. We conclude that μ is rectifiable. □

Even quite simple sets may, however, fail to be sparsely rectifiable, as the next example demonstrates. Indeed, even single Lipschitz curves may not be sparsely rectifiable, while being contained on such a set is equivalent to uniform rectifiability of Ahlfors-regular one-dimensional sets.

Example 1. Let us choose $h_j := 2^{-j}$ and $f_h(x) = h^{-2}\chi_Q(x/h)$ for $Q := [0,1]^2$. (Strictly speaking, $\{f_h\}$ is not normal per Definition 1. But this is only a matter of a translation or replacing $Q = [0,1]^2$. by the more unwieldy set $[-1/2, 1/2]^2$.) We then set $\Gamma_1 = [0,1] \times \{0\}$ and $\Gamma_2 = \{(x, g(x)) \mid x \in [0,1]\}$ for $g(x) = e^{-1/x}$, and study $\mu := \mathcal{H}^1 \llcorner (\Gamma_1 \cup \Gamma_2)$ on $\mathbb{R}^2$. See Figure 1(a) for a sketch.

Suppose $h \in (0,1)$ and let

$$A_h := \{(x,y) \mid x \in [0, 1-h],\, g(x+h) \le h,\, y \in [g(x+h) - h, 0]\}.$$



(a) A "simple" non-sparsely rectifiable set

(b) A "complex" sparsely rectifiable set

Figure 1: Examples of sparsely rectifiable and non-sparsely rectifiable sets

Then, whenever $(x, y) \in A_h$, both

$$\mathcal{H}^1(\Gamma_i \cap ((x+y) + hQ)) \geq h, \quad (i = 1, 2,).$$

Consequently, by the definition of f_h, we find that

$$(\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_h) \geq h^{-1}, \quad (i = 1, 2;\ (x, y) \in A_h).$$

If we set

$$\mathcal{G}_j^i := \{(\Gamma_1 \cup \Gamma_2 \setminus \Gamma_i) \cap ((x, y) + h_j Q) \mid (x, y) \in \mathbb{R}^2\},$$

we then have

$$h_j^{-1} \mathcal{L}^2(A_{h_j}) \leq \int_{A_{h_j}} (\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_{h_j})\, d(x, y) \leq \mathrm{Sp}_j(\mu; \mathcal{G}_j^i).$$

We want to show that A_h has too large measure for condition (3) to be satisfied, that is $h_j^{-1}\mathcal{L}^2(A_{h_j})$ does not sum to a finite quantity (for any sequence $h_j \searrow 0$).

For small enough h, we have

$$A_h \supset \{(x, y) \mid x \geq 0,\ g(x + h) \leq h/2,\ y \in [-h/2, 0]\}.$$

Since $g^{-1}(h) = -1/\log h$, we thus have (for small enough h)

$$h^{-1}\mathcal{L}^2(A_h) \geq h^{-1} \int_0^{g^{-1}(h/2)-h} h/2\, dx = (-1/\log(h/2) - h)/2.$$

We observe

$$\sum_{j=0}^{\infty} (-1/\log(h_j/2) - h_j) = \sum_{j=0}^{\infty} (1/(j+1) - 2^{-j}) = \infty.$$

Therefore $\sum_{j=0}^{\infty} \mathrm{Sp}_j(\mu; \mathcal{G}_j^i) = \infty$, $(i = 1, 2)$.

Finally, we observe that there do not exist families $\mathcal{G}_j$, $(j = 0, 1, 2, \ldots)$, of Lipschitz graphs covering $(\Gamma_1 \cup \Gamma_2) \cap ((x, y) + hQ)$ with bounded constant, so only Γ_1 or Γ_2 can be covered, as has been done above. To see this, one observes that for the Lipschitz constant to be bounded, there must exist $\alpha > 0$ such that any Lipschitz graph Γ covering a part Γ_1 has $|\langle z_\Gamma, (1, 0)\rangle| \geq \alpha$. But then either z_Γ is a tangent vector to Γ_2, or Γ_2 is occluded by Γ_1 when looking in the direction of z. Thus μ is not sparsely rectifiable with respect to the chosen system of rectangular nested functions.

As an element of justification for Definition 2, next we provide an example of a somewhat "complex" set that is sparsely rectifiable.

Example 2. Let $r_i := 2^{-i}$, and $\Gamma_i := \{1 - r_i\} \times [0, r_i]$, $(i = 0, 1, 2, \ldots)$. Also let $\Gamma' := [0, 1] \times \{0\}$. Finally, set $R := \Gamma' \cup \bigcup_{i=0}^{\infty} \Gamma_i$, as sketched in Figure 1(b). We claim that R is sparsely rectifiable with respect to $f_j(x) = h_j^{-2}\chi_Q(x/h_j)$, where $Q := [0, 1]^2$. Indeed, at every $(x, y) + h_jQ$, let us choose $\Gamma_j^{(x,y)}$ as $(\Gamma_i \bigcup \Gamma') \cap ((x, y) + h_jQ)$ for the smallest i such that $1 - r_i \geq x$. It follows that Γ' gives no contribution to (3) at all, so all we have to do is to calculate

$$Z_{i,j} := \int \mathcal{H}^1 \llcorner (\Gamma_i \setminus \Gamma_j^{(x,y)})(\tau_{(x,y)} f_j)\, d(x, y), \quad (i = 0, 1, 2, \ldots). \tag{6}$$

The term $\mathcal{H}^1 \llcorner (\Gamma_i \setminus \Gamma_j^{(x,y)})(\tau_{(x,y)} f_j)$ is non-zero only when $x + h_j \geq 1 - r_i$ and $x \leq 1 - r_{i-1}$. It follows that x is on an interval of length $h_j - r_i$, and $h_j \geq r_i$ (minding that $r_{i-1} - r_i = r_i$). For fixed x we may thus calculate that

$$\int (\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_j)\, dy = h_j^{-2} \int \int_y^{y+h} \chi_{[0,r_i]}(t)\, dt\, dy \leq r_i/h_j.$$

This gives the estimate

$$Z_{i,j} \leq \begin{cases} (h_j - r_i) r_i / h_j, & h_j \geq r_i, \\ 0, & \text{otherwise}, \end{cases}$$

for the contribution (6) of Γ_i, $(i = 0, 1, 2, \ldots)$, to (3). But $h_j \geq r_i$ means $i \geq -\log_2 h_j$, so summing the contributions of Γ_i for $i \geq -\log_2 h_j$, we obtain

$$\mathrm{Sp}_j(\mu) \leq \sum_{i=0}^{\infty} Z_{i,j} \leq \sum_{i \geq -\log_2 h} (h_j - r_i) r_i / h_j \leq \sum_{i \geq -\log_2 h_j} r_i \leq 2h_j.$$

Thus (3) holds. Moreover, it is possible to show that R is Ahlfors-regular in dimension 1, the maximum for the constant M for the upper bound being given at $(1, 0)$.

We next intend to show that sparse rectifiability implies uniform rectifiability for Ahlfors-regular sets. Towards this end, we recall the following definition (among many alternatives) from [7].

Definition 6. Suppose $K : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ is odd, i.e., $K(-x) = -K(x)$, and such that $\|x\|^{d+j} \|\nabla^j K(x)\| \in L^\infty(\mathbb{R}^m \setminus \{0\}$, $(j = 0, 1, 2, \ldots)$. An Ahlfors-regular set $E \subset \mathbb{R}^m$ (in dimension d) is said to be *uniformly rectifiable* if for any such K, the family of operators

$$T_\epsilon g(z) = \int_{E \setminus B(z,\epsilon)} K(z - y) g(y)\, d\mathcal{H}^d(y), \quad (\epsilon > 0),$$

determine bounded linear operators on $L^2(E)$ with the operator norm uniformly bounded in ϵ.

Lemma 1. *Let $E \subset \mathbb{R}^m$ be closed and Ahlfors-regular (in dimension d). Suppose there are constants $\theta \in (0, 1)$ and $L > 0$ such that for $\mathcal{H}^d$-a.e. $x \in E$ there exist $r_x > 0$ such that for all $r \in (0, r_x)$ one can find a Lipschitz graph Γ of Lipschitz factor at most L, satisfying*

$$\mathcal{H}^d(E \cap \Gamma \cap B(x, r)) \geq \theta \mathcal{H}^d(E \cap B(x, r)). \tag{7}$$

Then E is uniformly rectifiable in dimension d.

Proof. If we required that for *every* ball $B(x, r)$ with $x \in E$, $r > 0$, there existed a Lipschitz graph Γ of constant at most L such that (7) holds, then the present lemma would be just [7, Proposition 1.28] simplified to employ only Lipschitz graphs as Γ. The proof of [7, Proposition 1.28] is found in [6, Proposition III.3.2] (which is stated even more generally), and it turns out that in the proof one actually uses the stronger ("for every ball") assumptions only to produce a cover of E for which the slightly weaker assumptions of our lemma suffice. Therefore the proof of the present lemma follows directly from the proof of [6, Proposition III.3.2]. □

Proposition 2. *Sparse rectifiability of an Ahlfors-regular (in dimension $d \le m$) compact set $E \subset \mathbb{R}^m$ implies uniform rectifiability of E (in dimension d).*

Proof. We use Lemma 1. By Proposition 1, we may find a family $\{\Gamma_i\}_{i=0}^\infty$ of Lipschitz graphs such that $\mathcal{H}^d(E \setminus \bigcup_{i=0}^\infty \Gamma_i) = 0$. Moreover, by the proof of the proposition, thse graphs have have Lipschitz factor at most L, where L is the factor from Definition 2.

For an arbitrary $\epsilon \in (0, 1/2)$, we may then deduce (see the proof of [4, Theorem 2] for details) that for $\mathcal{H}^d$-a.e. $x \in E$, there exists $r_x > 0$ and $i \in \{0, 1, 2, \dots\}$ such that $\mathcal{H}^d(E\Delta\Gamma_i \cap B(x, r)) \le 2\epsilon r$ and $\mathcal{H}^d(E \cap B(x, r)) \ge 2(1 - \epsilon)r$ for $r \in (0, r_x)$. It follows that

$$\begin{aligned}\mathcal{H}^d(E \cap \Gamma_i \cap B(x, r)) &\ge \mathcal{H}^d(E \cap B(x, r)) - 2\epsilon r \\ &\ge (1 - 2\epsilon)/(1 - \epsilon)\mathcal{H}^d(E \cap B(x, r)), \quad (r \in (0, r_x)).\end{aligned}$$

Thus (7) holds at $\mathcal{H}^d$-a.e. $x \in E$ with $\theta = (1 - 2\epsilon)/(1 - \epsilon)$. This completes the proof. □

Remark 4. Observe that the sparse rectifiability condition is only used in the proof to ensure a bounded Lipschitz factor for the graphs $\{\Gamma_i\}$.

Example 3. The Cantor set K on the real line embedded into $\mathbb{R}^2$ as $K \times \{0\}$ is easily seen to be sparsely rectifiable; just take $\Gamma_j^x = S_j^x \cap ([0, 1] \times \{0\})$. Since K is not Ahlfors-regular from below in dimension 1, it is however not uniformly rectifiable.

Theorem 2. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\{\mu^i\}_{i=0}^\infty \in \mathrm{Sp}^d(\Omega, L, M)$ with*

$$\sup_{i=0,1,2,\dots} \mathrm{Sp}(\mu^i) + |\mu^i|(\Omega) < \infty.$$

Then any weak limit μ of $\{\mu^i\}_{i=0}^\infty$ satisfies $\mu \in \mathrm{Sp}^d(\Omega, L, M)$ and*

$$\mathrm{Sp}(\mu) + |\mu|(\Omega) \le \liminf_{i\to\infty} \mathrm{Sp}(\mu^i) + |\mu^i|(\Omega)$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $\mathcal{G}^i = \{\mathcal{G}^i_j\}_{j=0}^\infty$, $\mathcal{G}^i_j = \{\Gamma_j^{x,i} \mid x \in \mathbb{R}^m\}$, be such that

$$\mathrm{Sp}(\mu^i; \mathcal{G}^i) \le \mathrm{Sp}(\mu^i) + \epsilon, \quad (i = 0, 1, 2, \dots).$$

Then it suffices to show that

$$\mathrm{Sp}(\mu; \mathcal{G}) + |\mu|(\Omega) \le \liminf_{i\to\infty} \mathrm{Sp}(\mu^i; \mathcal{G}^i) + |\mu^i|(\Omega)$$

for some $\mathcal{G} = \{\mathcal{G}_j\}_{j=0}^\infty$, $\mathcal{G}_j = \{\Gamma_j^x \mid x \in \mathbb{R}^m\}$.

We use the shorthand notation $z_j^{x,i} = z_{\Gamma_j^{x,i}}$, and $g_j^{x,i} = g_{\Gamma_j^{x,i}}$. We may assume that $V_{\Gamma_j^{x,i}} = P^\perp_{z_j^{x,i}} B(x, h_j)$. This is because we may (see, e.g., [8]) extend $g_j^{x,i}$ from $V_{\Gamma_j^{x,i}}$ to the whole space $P^\perp_{z_j^{x,i}}$, without increasing the Lipschitz constant.

We may, moreover, assume that $\mu^i \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega)$, and $|\mu^i| \overset{*}{\rightharpoonup} \lambda \in \mathcal{M}(\Omega)$, where $\lambda \ge |\mu|$. The claim of the proposition now follows by an application of Fatou's inequality in (3), if we show for all $j = 0, 1, 2, \dots$ and almost all $x \in \mathbb{R}^m$ that

$$\liminf_{i\to\infty} \big|\mu^i \llcorner S_j^x \setminus \Gamma_j^{x,i}\big|(\tau_x f_j) \ge \big|\mu \llcorner S_j^x \setminus \Gamma_j^x\big|(\tau_x f_j) \tag{8}$$

for some Lipschitz graph Γ_j^x with constant at most L. Indeed, with $j = 0, 1, 2, \dots$ and $x \in \mathbb{R}^m$ fixed, we may for each $i = 0, 1, 2, \dots$, define a Lipschitz map $g_i : B(0, h_j) \subset \mathbb{R}^{m-1} \to \Gamma_j^x$ of constant at most

L by $g_i(v) = g_j^{x,i}(x + R_{z_j^{x,i}}(v, 0))$ with $R_z \in \mathbb{R}^{m\times m}$ the rotation matrix from $\mathbb{R}^{m-1} \times 0$ to $z^\perp$. Then, since Lipschitz maps of bounded constant are compact in the topology of pointwise convergence, we define Γ_j^x as the image of the pointwise limit g of a subsequence of $\{g_i\}_{i=0}^\infty$. Rotating the domain of g back on $z^\perp$ with z a limit of a further subsequence of $\{z_j^{x,i}\}_{i=0}^\infty$ will show that Γ_j^x is a Lipschitz graph.

Let us then write

$$\left|\mu^i \llcorner S_j^x \setminus \Gamma_j^{x,i}\right|(\tau_x f_j) = \left|\mu^i\right|(\tau_x f_j) - \left|\mu^i \llcorner \Gamma_j^{x,i}\right|(\tau_x f_j). \tag{9}$$

For almost all $x \in \mathbb{R}^m$, we have (as follows from, e.g., [1, Proposition 1.62])

$$|\mu^i|(\tau_x f_j) \to \lambda(\tau_x f_j). \tag{10}$$

Moreover, we have

$$\begin{aligned}\lambda(\tau_x f_j) &= (\lambda \llcorner S_j^x \setminus \Gamma_j^x)(\tau_x f_j) + (\lambda \llcorner \Gamma_j^x)(\tau_x f_j)\\ &\geq \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right|(\tau_x f_j) + (\lambda \llcorner \Gamma_j^x)(\tau_x f_j).\end{aligned} \tag{11}$$

On the other hand, any weak* limit $\widetilde{\lambda}$ of (a subsequence of) $|\mu^i| \llcorner \Gamma_j^{x,i}$ satisfies $\widetilde{\lambda} \leq \lambda \llcorner \Gamma_j^x$. Moreover, for a.e. $x \in \mathbb{R}^m$, we have $|\mu^i \llcorner \Gamma_j^{x,i}|(\tau_x f_j) \to \widetilde{\lambda}(\tau_x f_j)$. Thus, minding (9)–(11), we deduce

$$\begin{aligned}\liminf_{i\to\infty} \left|\mu^i \llcorner S_j^x \setminus \Gamma_j^{x,i}\right|(\tau_x f_j) &= \liminf_{i\to\infty} \Big(\left|\mu^i\right|(\tau_x f_j) - \left|\mu^i \llcorner \Gamma_j^{x,i}\right|(\tau_x f_j)\Big)\\ &\geq \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right|(\tau_x f_j) + (\lambda \llcorner \Gamma_j^x)(\tau_x f_j) - \limsup_{i\to\infty} \left|\mu^i \llcorner \Gamma_j^{x,i}\right|(\tau_x f_j)\\ &\geq \left|\mu \llcorner S_j^x \setminus \Gamma_j^{x,i}\right|(\tau_x f_j) + (\lambda \llcorner \Gamma_j^x)(\tau_x f_j) - \widetilde{\lambda}(\tau_x f_j)\\ &\geq \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right|(\tau_x f_j) \quad \text{for a.e. } x \in \mathbb{R}^m.\end{aligned}$$

But this is (8). Since upper Ahlfors regularity clearly holds for μ with constant M by the lower semi-continuity of $|\mu|(B(x, r))$ with respect to weak* convergence, we may conclude the proof. □

4. Poincaré-type inequalities

For an application of sparse rectifiability, soon to follow, we will need some Poincaré-type inequalities, which we next study. The following proposition can be found in, e.g., [11, Theorem 5.12.7].

Proposition 3. *Let $\Omega \subset \mathbb{R}^d$ be a connected domain with Lipschitz boundary, and μ a positive Radon measure on $\mathbb{R}^d$, that is upper Ahlfors regular with constant M in dimension $d-1$, and satisfies* $\operatorname{supp}\mu \subset \operatorname{cl}\Omega$. *Then there exists a constant $C_2 = C_2(\Omega)$, such that for each $u \in \mathrm{BV}(\Omega)$, we have*

$$\|u - \mu(u)/\mu(\Omega)\|_{L^1(\Omega)} \leq C_2 \frac{M}{\mu(\operatorname{cl}\Omega)} |Du|(\Omega).$$

Corollary 1. *Suppose $\Omega = B(0, r)$ in Proposition 3. Then there exists a constant $C_3 = C_3(d)$, independent of r, such that*

$$\|u - \mu(u)/\mu(\Omega)\|_{L^1(\Omega)} \leq r^d C_3 \frac{M}{\mu(\operatorname{cl}\Omega)} |Du|(\Omega), \quad (u \in \mathrm{BV}(\Omega)). \tag{12}$$

Suppose, in particular, that $\mu = \mathcal{L}^d \llcorner \Omega' \subset \Omega$ with $\mu(u) = 0$ and $\mathcal{L}^d(\Omega') \geq \rho r^d$. Then, for a constant $C_4 = C_4(d)$, we have

$$\|u\|_{L^1(\Omega)} \leq r\rho^{(1-d)/d} C_4 |Du|(\Omega). \tag{13}$$

Proof. We apply Proposition 3 on the domain $B(0,1)$ with $u_1(x) := u(rx)$ and $\mu_1(A) := \mu(rA)$. As $\mu_1(B(0,1)) = \mu(B(0,r))$, and the upper Ahlfors factor for μ_1 is at most Mr^{d-1}, a change of variables in the norms of u_1 gives (12).

As for the second result, we just have to approximate M. Elementary manipulations give

$$\mu(B(x,s)) \leq \min\{\omega_m s^d, \mathcal{L}^d(\Omega')\} \leq M s^{d-1}$$

for ω_m the volume of the unit ball in $\mathbb{R}^d$, and M defined by

$$M/\mathcal{L}^d(\Omega') = \left(\omega_m/\mathcal{L}^d(\Omega')\right)^{(d-1)/d} \leq (\rho^{-1}\omega_m)^{(d-1)/d} r^{1-d}.$$

Inserting this into (12) gives (13). □

Definition 7. Let $\mathcal{O}$ be a collection of domains in $\mathbb{R}^d$ (or any d-dimensional hyperplane of $\mathbb{R}^m$, $m \geq d$). Suppose there exist constants $\delta > 0$ and $K < \infty$, such that for each $\Omega \in \mathcal{O}$, there exists a ball $B_\Omega = B(x_\Omega, r_\Omega) \supset \Omega$ with $\mathcal{L}^d(\Omega) \geq \delta \mathcal{L}^d(B_\Omega)$ and an extension operator $T_\Omega : \mathrm{BV}(\Omega) \to \mathrm{BV}(B_\Omega)$ with $\|T_\Omega\| \leq K$. Then $\mathcal{O}$ is called a *collection of uniformly extensible domains.*

Lemma 2. *Suppose $\mathcal{O}$ is a collection of uniformly extensible domains. Then there exists a constant $C_5 = C_5(d,\delta,K)$ such that for each $\Omega \in \mathcal{O}$, the following holds: Let μ be positive Radon measure on $\mathbb{R}^d$, that is upper Ahlfors regular with constant M in dimension $d-1$, and satisfies* $\operatorname{supp}\mu \subset \operatorname{cl}\Omega$. *Then for each $u \in \mathrm{BV}(\Omega)$, we have*

$$\|u - \mu(u)/\mu(\Omega)\|_{L^1(\Omega)} \leq r_\Omega^d C_5 \frac{M}{\mu(\operatorname{cl}\Omega)} \|u\|_{\mathrm{BV}(\Omega)}.$$

In particular, if $\mu = \mathcal{L}^d \llcorner \Omega'$ with $\Omega' \subset \Omega$ and $\mathcal{L}^d(\Omega') \geq \rho r_\Omega^d$ with $\mu(u) = 0$, then for a constant $C_6 = C_6(d,\delta,K)$, we have

$$\|u\|_{L^1(\Omega)} \leq r_\Omega \rho^{(1-d)/d} C_6 \|u\|_{\mathrm{BV}(\Omega)}.$$

Proof. This lemma is a rather direct result of applying Corollary 1 to $T_\Omega u$. We just have to concede with having no bound for $|DT_\Omega u|(B_\Omega)$ directly, only that $|DT_\Omega u|(B_\Omega) \leq \|T_\Omega u\|_{\mathrm{BV}(B_\Omega)} \leq K\|u\|_{\mathrm{BV}(\Omega)}$. □

5. Convergence of total variation measures

First we recall and improve from [10] the following result. Then we study the relationship of η, introduced below, to sparse rectifiability. Throughout we assume that exactly the *same* nested sequence of functions $\{(f_j, \nu_j)\}_{j=0}^\infty$ is employed in the definition of $\mathrm{Sp}(\mu; \mathcal{G})$ and $\eta(\mu)$.

Theorem 3. *Let $\Omega \subset \mathbb{R}^m$ be an open set, and $\{(f_j, \nu_j)\}_{j=0}^\infty$ a normalised nested sequence of functions. For $\mu \in \mathcal{M}(\Omega)$, set*

$$\eta(\mu) := \sum_{j=0}^\infty \eta_j(\mu), \quad \text{where} \quad \eta_j(\mu) := |\mu|(\Omega) - \int |\mu(\tau_x f_j)|\, dx. \tag{14}$$

Suppose $\{\mu^i\}_{i=0}^\infty \subset \mathcal{M}(\Omega)$ weakly converges to $\mu \in \mathcal{M}(\Omega)$ with $\sup_i |\mu^i|(\Omega) + \eta(\mu^i) < \infty$. If also $|\mu^i| \overset{*}{\rightharpoonup} \lambda$, then $\lambda = |\mu|$, and $\eta_j(\mu^i) \to \eta_j(\mu)$ for each $j = 0, 1, 2, \ldots$. Moreover, the functional η is lower-semicontinuous with respect to the weak* convergence of $\{\mu^i\}_{i=0}^\infty$.*

Proof. Denote by S_f the set of (approximate) discontinuity points of f. Fubini's theorem and the fact that S_f is an $\mathcal{L}^m$-negligible Borel set, imply that $\int \lambda(S_{\tau_x f_j})\,dx = 0$. This shows that $\lambda(S_{\tau_x f_j}) = 0$ for a.e. $x \in \Omega$. As a consequence (see, e.g., [1, Proposition 1.62]), we have $\mu^i(\tau_x f_j) \to \mu(\tau_x f_j)$ for a.e. $x \in \mathbb{R}^m$. Likewise $|\mu^i|(\tau_x f_j) \to \lambda(\tau_x f_j)$ for a.e. $x \in \mathbb{R}^m$. Next we observe that $|\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)| \geq 0$, and

$$\lim_{i\to\infty} \int |\mu^i|(\tau_x f_j)\,dx = \lim_{i\to\infty} |\mu^i|(\mathbb{R}^m) = \lambda(\mathbb{R}^m) = \int \lambda(\tau_x f_j)\,dx = \int \lim_{i\to\infty} |\mu^i|(\tau_x f_j)\,dx.$$

An application of Fatou's lemma therefore yields

$$\begin{aligned}\lambda(\mathbb{R}^m) - \limsup_{i\to\infty} \int |\mu^i(\tau_x f_j)|\,dx &= \liminf_{i\to\infty} \left(\int |\mu^i|(\tau_x f_j)\,dx - \int |\mu^i(\tau_x f_j)|\,dx \right) \\ &\geq \int \liminf_{i\to\infty} \left(|\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)| \right)\,dx \\ &= \lambda(\mathbb{R}^m) - \int |\mu(\tau_x f_j)|\,dx,\end{aligned}$$

so that $\limsup_{i\to\infty} \int |\mu^i(\tau_x f_j)|\,dx \leq \int |\mu(\tau_x f_j)|\,dx$. By another application of Fatou's lemma on the sequence $\{x \mapsto |\mu^i(\tau_x f_j)|\}_{i=0}^\infty$, we thus get

$$\lim_{i\to\infty} \int |\mu^i(\tau_x f_j)|\,dx = \int |\mu(\tau_x f_j)|\,dx. \tag{15}$$

We stress that (15) holds because of the convergence $|\mu^i| \overset{*}{\rightharpoonup} \lambda$.

If we can show that, as claimed, $\lambda = |\mu|$, it follows immediately from (15) and the definition (14) that $\eta_j(\mu^i) \to \eta_j(\mu)$, showing that part of the claim of the lemma. Moreover, since the total variation $|\mu^i|(\Omega)$ is lower-semicontinuous with respect to weak* convergence, it follows from (15) that each η_j is lower-semicontinuous with respect to the simultaneous weak* convergence of $\{(\mu^i, |\mu^i|)\}_{i=0}^\infty$. Consequently also η is lower-semicontinuous with respect to the simultaneous convergence (by Fatou's lemma). However, assuming that $\{|\mu^i|\}_{i=0}^\infty$ does not converge, let us take a subsequence $\{\mu^{i_\ell}\}_{\ell=0}^\infty$ such that $\eta(\mu^{i_\ell}) \to \alpha := \liminf_{i\to\infty} \eta(\mu^i)$. Since $\sup_i |\mu^i|(\Omega) < \infty$, we may move to a further subsequence, unrelabelled, such that also $|\mu^{i_\ell}| \overset{*}{\rightharpoonup} \lambda$ for some $\lambda \in \mathcal{M}(\Omega)$. Since still $\eta(\mu^{i_\ell}) \to \alpha$, we deduce from the lower semicontinuity with respect to the simultaneous weak* convergence that $\alpha \geq \eta(\mu)$. This completes the proof of the claim that η is lower-semicontinuous with respect to weak* convergence of $\{\mu^i\}_{i=0}^\infty$ alone.

Returning to the proof of $\lambda = |\mu|$, observe that thanks to the fact that $\{(f_j, \nu_j)\}_{i=0}^\infty$ is a nested sequence of functions, $\{\eta_j(\mu)\}_{j=0}^\infty$ forms a decreasing sequence (for any $\mu \in \mathcal{M}(\Omega)$). Indeed, as $f_j(x) = \int f_{j+1}(x-y)\,d\nu_j(y)$ and $\nu_j(\mathbb{R}^m) = 1$ with $\nu_j \geq 0$, we have

$$\begin{aligned}\int |\mu(\tau_x f_j)|\,dx &= \int \left| \int \mu(\tau_{x+y} f_{j+1})\,d\nu_j(y) \right| dx \leq \int \int |\mu(\tau_{x+y} f_{j+1})|\,d\nu_j(y)\,dx \\ &= \int \int |\mu(\tau_{x+y} f_{j+1})|\,dx\,d\nu_j(y) = \int |\mu(\tau_x f_{j+1})|\,dx\end{aligned}$$

after a change of variables in the last step to eliminate y. Minding the definition (14), it follows from here that $\eta_j(\mu) \geq \eta_{j+1}(\mu)$.

To show $\lambda = |\mu|$, that is $|\mu^i| \overset{*}{\rightharpoonup} |\mu|$, we only have to show $|\mu^i|(\Omega) \to |\mu|(\Omega)$. To see the latter, we choose an arbitrary $\epsilon > 0$, and write

$$|\mu|(\Omega) - |\mu^i|(\Omega) = \eta_j(\mu) - \eta_j(\mu^i) + \int |\mu(\tau_x f_j)| - |\mu^i(\tau_x f_j)|\,dx. \tag{16}$$

Next we observe from the already proved lower semi-continuity of η and the bound $\sup_i \eta(\mu^i) =: K < \infty$ that $\eta(\mu) \leq K$ as well. Therefore, recalling that $\{\eta_j(\mu)\}_{j=1}^\infty$ and $\{\eta_j(\mu^i)\}_{j=1}^\infty$ for $i = 0, 1, \ldots$ are decreasing sequences, as shown above, it follows that by taking j large enough, we can ascertain that $\sup\{\eta_j(\mu), \eta_j(\mu^1), \eta_j(\mu^2), \ldots\} \leq \epsilon$. (Note that $\eta_j \geq 0$!) Employing this observation in (16), we find that

$$\big||\mu|(\Omega) - |\mu^i|(\Omega)\big| \leq 2\epsilon + \left| \int |\mu(\tau_x f_j)| - |\mu^i(\tau_x f_j)| \, dx \right|$$

for any large enough j and all i. The integral term tends to zero as $i \to \infty$ by (15). Therefore, we have

$$\lim_{i\to\infty} \big||\mu^i|(\Omega) - |\mu|(\Omega)\big| \leq 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof can be concluded. □

Remark 5. Since, by assumption, $\int f_j \, dx = 1$, we may alternatively write $\eta_j(\mu) = \int_{\mathbb{R}^m} |\mu|(\tau_x f_j) - |\mu(\tau_x f_j)| \, dx$.

Next we intend to derive bounds on $\eta(\mu)$ for sparsely rectifiable measures μ. We begin with a few definitions.

Definition 8. Suppose θ is a Borel function on a countably $\mathcal{H}^d$-rectifiable set $J \subset \mathbb{R}^m$, and $\widehat{\mathcal{G}}$ a family of Lipschitz d-graphs. We then set

$$\|\theta\|_{\mathrm{BV}(\widehat{\mathcal{G}})} = \sup \sum_{\Gamma_i} \|\theta \circ g_{\Gamma_i}\|_{\mathrm{BV}(\Gamma_i)},$$

where the supremum is taken over all finite *disjoint* sub-collections $\{\Gamma_1, \ldots, \Gamma_N\} \subset \widehat{\mathcal{G}}$, $(N \geq 1)$.

Definition 9. Let $\mu \in \mathrm{Sp}^d(\Omega, L, M)$ with Definition 2 satisfied by the families $\mathcal{G}^\mu = \{\mathcal{G}^\mu_j\}_{j=0}^\infty$, $\mathcal{G}^\mu_j = \{\Gamma^x_j \mid x \in \mathbb{R}^m\}$ of Lipschitz graphs. We then denote

$$\widehat{\mathcal{G}}^\mu_j := \{g^x_j(V^x_j \cap P^\perp_{z^x_j} B(x, h_j)) \mid x \in \mathbb{R}^m,\, \Gamma^x_j \cap B(x, h_j) \neq \emptyset\}, \quad (j = 0, 1, 2, \ldots),$$

and $\widehat{\mathcal{G}}^\mu := \{\widehat{\mathcal{G}}^\mu_j\}_{j=0}^\infty$, as well as

$$\mathcal{O}^\mu := \bigcup_{j=0}^\infty \{V^x_j \cap P^\perp_{z^x_j} B(x, h_j) \mid x \in \mathbb{R}^m,\, \Gamma^x_j \cap B(x, h_j) \neq \emptyset\}.$$

Remark 6. Let us denote $\Omega^x_j = V^x_j \cap P^\perp_{z^x_j} B(x, h_j)$. Instead of considering the sub-graphs

$$g^x_j(\Omega^x_j) \subset g^x_j(V^x_j) = \Gamma^x_j,$$

we could simply require that

$$P^\perp_{z^x_j} \Gamma^x_j \subset P^\perp_{z^x_j} B(x, h_j),$$

as such a requirement has no no effect on Definition 2 holding for μ: only $\Gamma^x_j \cap B(x, h_j)$ matters in the expression of $\mathrm{Sp}_j(\mu; \mathcal{G}_j)$, so that $\mathrm{Sp}_j(\mu; \mathcal{G}^\mu_j) = \mathrm{Sp}_j(\mu; \widehat{\mathcal{G}}^\mu_j)$.

The reason for the restriction of the domain is, firstly, to ensure that the radii $r_{\Omega^x_j}$ in Definition 7 can be taken small, so that functions in $\mathrm{BV}(\Omega^x_j)$ can with uniformly bounded constants be extended to $P^\perp_{z^x_j} B(x, h_j) \supset \Omega^x_j$. Secondly, the restriction ensures that $g^x_j(\Omega^x_j) \subset B(x, (2L+4)h_j)$ whenever $B(x, h_j) \cap \Gamma^x_j \neq \emptyset$. We will need this in the following proposition, for which we recall the following constants.

- L is an upper bound on the Lipschitz constant of the graphs $\Gamma^x_j \in \bigcup_{j=0}^\infty \widehat{\mathcal{G}}_j$,
- α is the regularity constant for the maps $\{f_j\}_{j=0}^\infty$ from Definition 1, and
- δ and K are the uniform extensibility constants of $\mathcal{O}^\mu$.

Proposition 4. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\mu = \theta\mathcal{H}^d \llcorner J \in \mathrm{Sp}^d(\Omega, L, M)$, and $\mathcal{O}^\mu$ is uniformly extensible per Definition 7. Then*

$$\eta_j(\mu) \le C_7 h_j \|\theta\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j^\mu)} + \mathrm{Sp}_j(\mu; \mathcal{G}_j^\mu) \tag{17}$$

for some constant $C_7 = C_7(L, m, d, \delta, K, \alpha)$. In particular, if $\sum_{j=0}^\infty h_j < \infty$, then

$$\eta(\mu) \le C_8\big(\sup_{j=0,1,2,\ldots} \|\theta\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j^\mu)} + \mathrm{Sp}(\mu; \mathcal{G}^\mu)\big)$$

for $C_8 = C_8(L, m, d, \delta, K, \alpha, \sum h_j)$.

Proof. Let us simply assume, without loss of generality as discussed in Remark 6, that

$$P_{z_j^x}^\perp \Gamma_j^x \subset P_{z_j^x}^\perp B(x, h_j), \quad (x \in \mathbb{R}^m;\ j = 0, 1, 2, \ldots). \tag{18}$$

as well as

$$\Gamma_j^x \cap B(x, h_j) \neq \emptyset, \quad (x \in \mathbb{R}^m;\ j = 0, 1, 2, \ldots).$$

Then we may simply write

$$\widehat{\mathcal{G}}_j^\mu := \mathcal{G}_j^\mu, \quad (j = 0, 1, 2, \ldots),$$

and

$$\mathcal{O}^\mu := \{V_\Gamma \mid \Gamma \in \bigcup_{j=0}^\infty \widehat{\mathcal{G}}_j^\mu\}.$$

Let then $j \in \{0, 1, 2, \ldots\}$ be fixed for the moment. By writing $\theta = \theta^+ - \theta^-$, where $\theta^\pm \ge 0$, we deduce

$$\begin{aligned} \eta_j(\mu) &= \int |\mu|(\tau_x f_j) - |\mu(\tau_x f_j)|\, dx \\ &= 2\int \min\left\{\int_J \theta^+ \tau_x f_j\, d\mathcal{H}^d, \int_J \theta^- \tau_x f_j\, d\mathcal{H}^d\right\} dx. \end{aligned} \tag{19}$$

Writing $J = (J \cap \Gamma_j^x) \cup (J \setminus \Gamma_j^x)$, we get

$$\eta_j(\mu)/2 \le \int \min\left\{\int_{\Gamma_j^x} \theta^+ \tau_x f_j\, d\mathcal{H}^d, \int_{\Gamma_j^x} \theta^- \tau_x f_j\, d\mathcal{H}^d\right\} dx + \int \big|\mu \llcorner S_j^x \setminus \Gamma_j^x\big|(\tau_x f_j)\, dx. \tag{20}$$

Since the minimum is non-zero only if both $\theta^+|S_j^x \neq 0$ and $\theta^-|S_j^x \neq 0$, only points x in the set

$$Z_j := \{x \in \mathbb{R}^m \mid 0 \in \operatorname{conv} \theta(\Gamma_j^x),\ \Gamma_j^x \cap B(x, h_j) \neq \emptyset\}$$

contribute to the first integral in (20). Applying (3), we thus obtain

$$\begin{aligned} \eta_j(\mu)/2 &\le \int_{Z_j} \min\left\{\int_{\Gamma_j^x} \theta^+ \tau_x f_j\, d\mathcal{H}^d, \int_{\Gamma_j^x} \theta^- \tau_x f_j\, d\mathcal{H}^d\right\} dx + \mathrm{Sp}_j(\mu; \widehat{\mathcal{G}}_j^\mu) \\ &\le \alpha^{-1} h_j^{-m} \int_{Z_j} \min\left\{\int_{\Gamma_j^x} \theta^+\, d\mathcal{H}^d, \int_{\Gamma_j^x} \theta^-\, d\mathcal{H}^d\right\} dx + \mathrm{Sp}_j(\mu; \widehat{\mathcal{G}}_j^\mu). \end{aligned} \tag{21}$$

In the final step we have used the regularity of $\{f_j\}_{j=0}^\infty$, i.e., $f_j \le \alpha^{-1} h_j^{-m} \chi_{B(0,h_j)}$.

Next we set $B_j := B(0, (2L+4)h_j)$, and apply the Besicovitch covering theorem on the family $\{B_j + x \mid x \in Z_j\}$ to find finite collections $F_j^1, \ldots, F_j^{c_m} \subset Z_j$ satisfying $\sum_{x \in F_j^i} \tau_x \chi_{B_j} \le 1$, $(i =$

$1,\ldots,c_m$), and $\sum_{x\in F_j}\tau_x\chi_{B_j}\geq\chi_{Z_j}$. Here $F_j:=\bigcup_{i=1}^{c_m}F_j$ and c_m is a constant dependent on the dimension m alone. Applying the cover F_j in (21), and denoting $\Gamma_j^x(\theta)=\int_{\Gamma_j^x}\theta\,d\mathcal{H}^d$, we may write

$$\begin{aligned}\eta_j(\mu)/2&\leq\alpha^{-1}h_j^{-m}\int_{B_j}\sum_{x\in(F_j+y)\cap Z_j}\min\{\Gamma_j^x(\theta^+),\Gamma_j^x(\theta^-)\}\,dy+\mathrm{Sp}_j(\mu;\widehat{\mathcal{G}}_j^\mu)\\&\leq\frac{C_9}{\mathcal{L}^m(B_j)}\int_{B_j}\sum_{x\in(F_j+y)\cap Z_j}\min\{\Gamma_j^x(\theta^+),\Gamma_j^x(\theta^-)\}\,dy+\mathrm{Sp}_j(\mu;\widehat{\mathcal{G}}_j^\mu)\end{aligned}\tag{22}$$

for some constant $C_9=C_9(\alpha,m,L)$. By the definition of F_j as $\bigcup_{i=1}^{c_m}F_j^i$, it follows that to bound $\eta_j(\mu)$, it suffices to show that there exists $C_{10}=C_{10}(d,\delta,K,L)$ such that

$$\sum_{x\in(F_j^i+y)\cap Z_j}\min\{\Gamma_j^x(\theta^+),\Gamma_j^x(\theta^-)\}\leq C_{10}h_j\|\theta\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j)}\tag{23}$$

for $\mathcal{L}^m$-a.e. $y\in B_j$ and all $i\in\{1,\ldots,c_m\}$.

To begin the proof of (23), we observe that $\mathcal{J}_d(\nabla g_j^x(v))\leq\sqrt{\binom{m}{d}}L^d=:C_{11}$. (Indeed, any $d\times d$ minor B of $\nabla g_j^x(v)$ satisfies $\det B\leq L^d$, due to the maximal eigenvalue being at most L. Then we apply the Cauchy-Binet formula, found in, e.g., [2].) Thus the area formula yields

$$\Gamma_j^x(\theta^\pm)=\int_{\Gamma_j^x}\theta^\pm\,d\mathcal{H}^d=\int_{V_j^x}(\theta^\pm\circ g_j^x)\mathcal{J}_d(\nabla g_j^x)\,dv\leq C_{11}\int_{V_j^x}\theta^\pm\circ g_j^x\,dv.\tag{24}$$

Let us momentarily fix $x\in Z_j$, and set $V=V_j^x$, $\widetilde{\theta}^\pm=\theta^\pm\circ g_j^x$, and $\widetilde{\theta}=\theta\circ g_j^x$. We intend to apply Lemma 2. Toward this end, we set $\mu^\pm:=\mathcal{L}^d\llcorner(V\setminus\operatorname{supp}\widetilde{\theta}^\mp)$. Then $\mu^+(V)+\mu^-(V)\geq\mathcal{L}^d(V)$, so with the constant δ from Definition 7, we have

$$\max\{\mu^+(V),\mu^-(V)\}\geq\mathcal{L}^d(V)/2\geq(\delta/2)\mathcal{L}^d(B_V)\geq(\delta\omega_d/2)r_V^d$$

for r_V as in Definition 7. Since $\mu^\pm(\widetilde{\theta}^\pm)=0$, we may apply Lemma 2, to get either

$$\|\widetilde{\theta}^+\|_{L^1(V)}\leq r_VC_{12}\|\widetilde{\theta}^+\|_{\mathrm{BV}(V)}\quad\text{or}\quad\|\widetilde{\theta}^-\|_{L^1(V)}\leq r_VC_{12}\|\widetilde{\theta}^-\|_{\mathrm{BV}(V)}$$

for a constant $C_{12}=C_{12}(d,\delta,K)$. As $\|\widetilde{\theta}^\pm\|_{\mathrm{BV}(V)}\leq\|\widetilde{\theta}\|_{\mathrm{BV}(V)}$, by the definition of $\theta^\pm$, this gives

$$\min\{\|\widetilde{\theta}^+\|_{L^1(V)},\|\widetilde{\theta}^-\|_{L^1(V)}\}\leq r_VC_{12}\|\widetilde{\theta}\|_{\mathrm{BV}(V)}.$$

The crucial point here is that r_V is "small". Indeed, we may assume $r_V\leq h_j$ due to (18). We thus obtain

$$\min\{\|\theta^+\circ g_j^x\|_{L^1(V_j^x)},\|\theta^-\circ g_j^x\|_{L^1(V_j^x)}\}\leq h_jC_{12}\|\theta\circ g_j^x\|_{\mathrm{BV}(V_j^x)}.\tag{25}$$

Next, we observe that with all $j\in\{0,1,2,\ldots\}$, $i\in\{1,\ldots,c_m\}$, and $y\in B_j$ fixed, the graphs $\{\Gamma_j^x\mid x\in(y+F_j^i\cap Z_j)\}$ are disjoint. This follows from the balls $x+B_j=B(x,(2L+4)h_j)$, $(x\in y+F_j^i)$, being disjoint by construction, and from $\Gamma_j^x\subset x+B_j$, that holds due to assumption (18). Combining (25) with (24) thus finally yields

$$\begin{aligned}\sum_{x\in(F_j^i+y)\cap Z_j}\min\{\Gamma_j^x(\theta^+),\Gamma_j^x(\theta^-)\}&\leq C_{11}C_{12}h_j\sum_{x\in(F_j^i+y)\cap Z_j}\|\theta\circ g_j^x\|_{\mathrm{BV}(V_j^x)}\\&\leq C_{11}C_{12}h_j\|\theta\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j)}.\end{aligned}\tag{26}$$

To conclude the proof of the lemma, we only have to observe that (26) yields (23). □

Corollary 2. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $\sum_j h_j < \infty$. Suppose $\{\mu^i\}_{i=0}^{\infty} \subset \mathrm{Sp}^d(\Omega, L, M)$ and that $\bigcup_{i=0}^{\infty} \mathcal{O}^{\mu^i}$ is uniformly extensible. Provided that*

$$\sup_{i,j} \left(\|\theta^i\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j^{\mu^i})} + \mathrm{Sp}(\mu^i; \mathcal{G}_j^{\mu^i}) \right) < \infty,$$

then any weak limit μ of a subsequence of $\{\mu^i\}_{i=0}^{\infty}$, unrelabelled, satisfies $\mu \in \mathrm{Sp}^d(\Omega, L, M)$ and $|\mu^i|(\Omega) \to |\mu|(\Omega)$.*

If additionally

$$\sup_i \mathrm{Sp}_j(\mu^i; \mathcal{G}_j^{\mu^i}) \leq v(j) \text{ for some } v \in L^1(\mathbb{N}),$$

then $\eta(\mu^i) \to \eta(\mu)$.

Proof. The claim on the convergence of $|\mu^i|(\Omega)$ to $|\mu|(\Omega)$ is an easy consequence of Proposition 4, Theorem 3, and Theorem 2. That $\eta(\mu^i) \to \eta(\mu)$ follows from the already known lower-semicontinuity of η and the limsup variant of Fatou's lemma, after the facts that, by Theorem 3, $\eta_j(\mu^i) \to \eta_j(\mu)$, and, by (17), $\eta_j(\mu^i) \leq C_{13} h_j + v(j)$, $(i = 0, 1, 2, \ldots)$, where $j \mapsto C_{13} h_j + v(j)$ is integrable. □

6. Compactness in $[\mathrm{SBV}(\Omega)]^k$

We will now provide some compactness and other convergence results in $[\mathrm{SBV}(\Omega)]^k$ following from sparse rectifiability. All of the results hold with exactly the same proofs in $\mathrm{SBD}(\Omega)$, when the gradient Du is replaced by the symmetricised gradient $Eu := (Du + (Du)^T)/2$, and ∇u and $D^j u$ are replaced, respectively, by the density $\mathcal{E}u$ of the absolutely continuous part of Eu, and by jump part $E^j u$. See Temam [9] for basics on functions of bounded deformation, and Ambrosio et al. [1] for SBD.

We need to work with vector-valued measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{k \times m})$ now. The results of Section 3 can readily be extended to this situation with no changes in proofs or definitions, but for concreteness we work through the following definition.

Definition 10. For $\mu = (\mu_{i,j}) \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k \times m}$, we denote $\mathrm{Sp}(\mu) = \sum_{i=1}^{k} \sum_{j=1}^{m} \mathrm{Sp}(\mu_{i,j})$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $\{u^i\}_{i=0}^{\infty} \subset [\mathrm{SBV}(\Omega)]^k$. Suppose $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing with $\lim_{t \to \infty} \psi(t)/t = \infty$. If each $D^j u^i \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k \times m}$, $(i = 0, 1, 2, \ldots)$, and*

$$\sup_i \|u^i\|_{L^1(\Omega)} + \int \psi(\nabla u^i(x))\, dx + |D^j u^i|(\Omega) + \mathrm{Sp}(D^j u^i) < \infty, \tag{27}$$

there then exists $u \in [\mathrm{SBV}(\Omega)]^k$ with $D^j u \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k \times m}$ and a subsequence, unrelabelled, such that

$$u^i \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^k), \tag{28}$$

$$\nabla u^i \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega; \mathbb{R}^{k \times m}), \tag{29}$$

$$D^j u^i \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(\Omega; \mathbb{R}^{k \times m}), \quad \text{and} \tag{30}$$

$$\mathrm{Sp}(D^j u) \leq \liminf_{i \to \infty} \mathrm{Sp}(D^j u^i). \tag{31}$$

Proof. Let us denote by K the supremum on the left side of (27). We then deduce from (27) that

$$\sup_i \|u^i\|_{L^1(\Omega)} + |Du^i|(\Omega) < \infty.$$

Moving to a subsequence, unrelabelled, we may thus assume that $u^i \rightharpoonup u$ weakly in $[\mathrm{BV}(\Omega)]^k$ for some $u \in [\mathrm{BV}(\Omega)]^k$. This gives (28). Moreover, because $\{\nabla u^i\}_{i=0}^\infty$ is an equi-integrable family, we have the existence of some $w \in L^1(\Omega; \mathbb{R}^{k\times m})$, such that for a further unrelabelled subsequence, $\nabla u^i \rightharpoonup w$ weakly in $L^1(\Omega; \mathbb{R}^{k\times m})$. Still, selecting another subsequence, we find from Theorem 2 that $D^j u^i \overset{*}{\rightharpoonup} \lambda$ for some $\lambda \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k\times m}$ with $\mathrm{Sp}(\lambda) \leq \liminf_{i\to\infty} \mathrm{Sp}(D^j u^i)$. Minding that $\nabla u^i \mathcal{L}^m + D^j u^i = Du^i$ and $Du^i \overset{*}{\rightharpoonup} Du$ by the weak convergence of $\{u^i\}_{i=0}^\infty$ in $[\mathrm{BV}(\Omega)]^k$, we therefore have

$$w\mathcal{L}^m + \lambda = Du = \nabla u \mathcal{L}^m + D^j u + D^c u. \tag{32}$$

Since $\lambda \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k\times m}$, Proposition 1 shows that the measure λ is concentrated on a $\mathcal{H}^{m-1}$ rectifiable set J. This gives $w = \nabla u$, showing (29). According to [1], the Cantor part $D^c u$ vanishes on any Borel set B that is σ-finite with respect to $\mathcal{H}^{m-1}$. In particular $D^c u \llcorner J = 0$. Hence, by (32), $\lambda = D^j u$ and $D^c u = 0$. This shows that $u \in [\mathrm{SBV}(\Omega)]^k$ as well as (30) and (31), thus completing the proof. □

Remark 7. Theorem 4 is complementary to the SBV compactness theorem of Ambrosio (Theorem 1), or in the SBD case, the analogue due to Bellettini et al. [3]. While our result requires bounds on the sparse rectifiability of $D^j u^i$, we do not require $\mathcal{H}^{m-1}(J_{u^i})$ to be bounded, unlike does Theorem 1.

We now provide a corollary with stronger convergence for the jump part of the gradient. Choosing T below as the trace operator, $T = \mathrm{Tr}$, we get the convergence of $\mathrm{Div}^j u^i = \mathrm{Tr}\, E^j u^i$ in total variation.

Corollary 3. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $\{u^i\}_{i=0}^\infty \subset [\mathrm{SBV}(\Omega)]^k$. Suppose $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$, and $T : \mathbb{R}^{k\times m} \to \mathbb{R}$ a bounded linear operator. If each $D^j u^i \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k\times m}$, $(i = 0, 1, 2, \dots)$, and*

$$\sup_i \|u^i\|_{L^1(\Omega)} + \int \psi(\nabla u^i(x))\, dx + |D^j u^i|(\Omega) + \mathrm{Sp}(D^j u^i) + \eta(TD^j u^i) < \infty, \tag{33}$$

then there exists $u \in [\mathrm{SBV}(\Omega)]^k$ with $D^j u \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{k\times m}$, and a subsequence, unrelabelled, such that (28)–(31) hold along with

$$|TD^j u^i|(\Omega) \to |TD^j u|(\Omega). \tag{34}$$

Proof. Theorem 4 shows that (28)–(31) hold along with $TD^j u^i \overset{*}{\rightharpoonup} TD^j u$ weakly* on $\mathcal{M}(\Omega)$. Now (34) follows from Theorem 3. □

Remark 8. Of course, provided that $\bigcup_{i=0}^\infty \mathcal{O}^{TD^j u^i}$ is uniformly extensible and $\sum_{j=0}^\infty h_j < \infty$, we can apply Corollary 2. This allows us to replace $\eta(TD^j u^i)$ in (33) by $\sup_j \|\theta^i\|_{\widehat{\mathcal{G}}_j^{TD^j u^i}}$, where θ^i is such that $TD^j u^i = \theta^i \mathcal{H}^{m-1} \llcorner J^i$. Further variants yet of Corollary 2 are possible, along with closely-related lower-semicontinuity results (compare, e.g., [3]), thanks to the lower-semicontinuity results contained in Theorem 2 and Theorem 3.

We base the next demonstrative result on Theorem 1 instead of Theorem 4. This is due to existing approximation results that ensure the convergence of $\mathcal{H}^{m-1}(J_{u^i})$, while not much is yet known of the approximability of $\mathrm{Sp}(D^j u^i)$.

Definition 11. We denote $L^\infty_M(\Omega; \mathbb{R}^k) := \{u \in L^\infty(\Omega; \mathbb{R}^k) \mid \|u\|_{L^\infty(\Omega)} \leq M\}$.

Proposition 5. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded and $M > 0$. Let $F : L^1(\Omega) \to \mathbb{R}$ be continuous and bounded from below, $\psi : [0, \infty) \to [0, \infty)$ non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$, and $T : \mathbb{R}^{m\times m} \to \mathbb{R}$ a bounded linear operator. Set*

$$J(u) := F(u) + \int \psi(\nabla u(x))\, dx + \mathcal{H}^{m-1}(J_u) + \eta(TD^j u).$$

Then we have the following.

1. *J admits a minimiser in $[\mathrm{SBV}(\Omega)]^k \cap L^\infty_M(\Omega; \mathbb{R}^k)$.*
2. *Suppose that $\{u, u^0, u^1, u^2, \ldots\} \subset [\mathrm{SBV}(\Omega)]^k \cap L^\infty_M(\Omega; \mathbb{R}^k)$ satisfy*

$$u^i \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^m), \tag{35}$$

$$\nabla u^i \to \nabla u \text{ strongly in } L^1(\Omega; \mathbb{R}^{k\times m}), \quad \text{and} \tag{36}$$

$$\mathcal{H}^{m-1}(J_{u^i}) \to \mathcal{H}^{m-1}(J_u). \tag{37}$$

Suppose, moreover, that $\mu^i := TD^j u^i \in \mathrm{Sp}^d(\Omega, L, M)$, $(i = 0, 1, 2, \ldots)$, with $\bigcup_{i=0}^\infty \mathcal{O}^{\mu^i}$ uniformly extensible, $\sum_{j=0}^\infty h_j < \infty$, and that

$$\sup_{i,j} \|\theta^i\|_{\mathrm{BV}(\widehat{\mathcal{G}}_j^{\mu^i})} < \infty,$$

as well as

$$\sup_i \mathrm{Sp}_j(\mu^i; \widehat{\mathcal{G}}_j^{\mu^i}) \le v(j) \text{ for some } v \in L^1(\mathbb{N}).$$

Then there exists a subsequence $\{u^{i_j}\}_{j=0}^\infty$ of $\{u^i\}_{i=0}^\infty$ with $J(u^{i_j}) \to J(u)$.

Proof. That J admits a minimiser follows from Theorem 1 and Theorem 3 by a standard argument, minding that $u \in L^\infty_M(\Omega)$ and $\mathcal{H}^{m-1}(J_u)$ bound $|D^j u|(\Omega)$.

As for the convergence claim, it follows from Corollary 2 that $\eta(TD^j u^{i_j}) \to \eta(TD^j u)$ for a subsequence $\{u^{i_j}\}_{j=0}^\infty$ of $\{u^i\}_{i=0}^\infty$. Combined with (35)–(37) and the continuity of F and ψ, this immediately yields $J(u^{i_j}) \to J(u)$. □

Remark 9. It follows from the work of Cortesani and Toader [5] that given $u \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$, there then exist $u^i \in H^1(\Omega \setminus J^i) \cap L^\infty_M(\Omega)$, with J^i a finite union of C^1 surfaces such that (35)–(37) hold. A similar result is provided by Chambolle [4] in the SBD case. It is ongoing research how we can satisfy the additional assumptions of Proposition 5 or otherwise ensure $\eta(TD^j u^i) \to \eta(TD^j u)$.

One final remark

Remark 10. All of our results continue to hold with simplified proofs for the following "discrete" versions of η and the sparse rectifiability condition. Instead of integrating over all $x \in \mathbb{R}^m$ and requiring $\int \tau_x f_j\, dx \equiv 1$, we would then sum over x on finite grids F_j such that $\sum_{x\in F_j} \tau_x f_j \equiv 1$. Our quantities of interest would then be

$$\widetilde{\mathrm{Sp}}_j(\mu; \mathcal{G}_j) := \sum_{x\in F_j} \left|\mu \llcorner S_j^x \setminus \Gamma_j^x\right|(\tau_x f_j),$$

and

$$\widetilde{\eta}_j(\mu) := |\mu|(\Omega) - \sum_{x\in F_j} |\mu(\tau_x f_j)|.$$

In this case we need to assume the lower semi-continuity of each f_j, but the Besicovitch covering arguments in the proofs not be needed, as we are already working with finite sets.

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