A method for weighted projections to the positive definite cone

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Abstract

We study the numerical solution of the problem $\min_{\mathbf{X}\geq\mathbf{0}} \|\mathbf{B}\mathbf{X}-\mathbf{c}\|_2$, where $\mathbf{X}\in\mathcal{S}_m$ is a symmetric square matrix, and $\mathbf{B}:\mathcal{S}_m\to\mathbb{R}^N$ is a linear operator, such that $\mathbf{M}:=\mathbf{B}^*\mathbf{B}$ is invertible. With ρ the desired fractional duality gap, and κ the condition number of $\mathbf{M}$, we prove $O(\sqrt{m}\kappa^2\log\rho^{-1})$ iteration complexity for a simple primal-dual interior point method directly based on those for linear programs with semi-definite constraints. We do not, however, require the numerically expensive scalings inherent in these methods to force fast convergence. For low-dimensional problems ($m\leq 10$), our numerical experiments indicate excellent performance and only a very slowly growing dependence of the convergence rate on κ. While our algorithm requires somewhat more iterations than existing interior point methods, the iterations are cheaper. This gives better computational times.

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1. Introduction

Let $\mathbf{B}:\mathcal{S}_m\to\mathbb{R}^N$ be a linear operator on symmetric $m\times m$ real matrices. Assume that $\mathbf{M}:=\mathbf{B}^*\mathbf{B}$ is invertible. Given $\mathbf{c}\in\mathbb{R}^N$, we study the solution of the problem

$$\min_{\mathbf{X}\in\mathcal{S}_m^+} \|\mathbf{B}\mathbf{X}-\mathbf{c}\|_2, \tag{P}$$

with $\mathcal{S}_m^+:=\{\mathbf{X}\in\mathcal{S}_m \mid \mathbf{X}\geq\mathbf{0}\}$ denoting the set of positive semi-definite symmetric $m\times m$ matrices. Setting $\mathbf{C}:=\mathbf{B}^*\mathbf{c}$, this is an instance of the quadratic optimisation problem with positive semi-definite constraints (quadratic SDP)

$$\min_{\mathbf{X}\in\mathcal{S}_m^+} \frac{1}{2}\langle\mathbf{X},\mathbf{M}\mathbf{X}\rangle-\langle\mathbf{C},\mathbf{X}\rangle, \tag{Q}$$

where we assume that the linear operator $\mathbf{M}:\mathcal{S}_m\to\mathcal{S}_m$ is self-adjoint and positive definite. More generally, both (P) and (Q) are instances of *semi-definite linear complementarity problems* (SDLCPs), discussed in, e.g., [21, 20, 26].

To motivate the study of the instance (P), we first observe that if $\mathbf{B}\mathbf{X}=\mathrm{vec}(\mathbf{X})$ is the vectorisation of the matrix $\mathbf{X}$, then $\|\mathbf{B}\mathbf{X}-\mathbf{c}\|_2=\|\mathbf{X}-\mathbf{C}\|_F$, where $\mathbf{C}:=\mathrm{vec}^{-1}(\mathbf{c})$. Thus the solution $\hat{\mathbf{X}}$ of (P) is simply the projection of $\mathbf{C}\in\mathcal{S}_m$ to the positive semi-definite cone per the Frobenius norm. As a second source of motivation, we introduce an application from diffusion tensor imaging (DTI) [3, 18, 41]. A diffusion tensor field $\mathbf{u}\in L^1(\Omega;\mathcal{S}_3)$ on a domain $\Omega\subset\mathbb{R}^3$ is determined by the Stejskal-Tanner equation

$$a_i(x)=a_0(x)\exp(-\langle\mathbf{b}_i\otimes\mathbf{b}_i,\mathbf{u}(x)\rangle), \quad (x\in\Omega), \tag{1.1}$$

from multiple diffusion weighted MRI (magnetic resonance imaging) measurements $a_i\in L^1(\Omega)$, for varying diffusion gradients $\mathbf{b}_i\in\mathbb{R}^3$, $(i=1,\ldots,N)$, as well as the zero gradient $\mathbf{b}_0=0$. When there are

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at least six measurements, such that the matrices $\mathbf{B}_i := \mathbf{b}_i \otimes \mathbf{b}_i$ form a (necessarily non-orthogonal!) basis of $\mathcal{S}_3$, then $\mathbf{u}(x)$ can be determined by linear least-squares from (1.1). Namely, at each $x \in \Omega$, we solve

$$\min_{\mathbf{X} \in \mathcal{S}_3} \sum_{i=1}^{N} (\langle \mathbf{B}_i, \mathbf{X} \rangle - c_i)^2,$$

where $c_i := -\log(a_i(x)/a_0(x))$. Writing $\mathbf{BX} := (\langle \mathbf{B}_1, \mathbf{X} \rangle, \ldots, \langle \mathbf{B}_N, \mathbf{X} \rangle)$ and $\mathbf{c} := (c_1, \ldots, c_N)$, this problem has the form

$$\min_{\mathbf{X} \in \mathcal{S}_m} \|\mathbf{BX} - \mathbf{c}\|_2. \tag{1.2}$$

The solution is, of course, $\mathbf{X} = \mathbf{M}^{-1}\mathbf{B}^*\mathbf{c}$, provided $\mathbf{M}$ is invertible. This is how the diffusion tensor $\mathbf{u}(x) = \mathbf{X}$ is conventionally solved [3]. The only difference between (1.2) and (P) is that in the former there is no semi-definiteness constraint. But, *non-positive-definite diffusion-tensors are non-physical*, and the solution of (1.2) might not be positive definite, or even positive semi-definite. This is why it should be better, from an application point of view, to solve the problem (P) instead of (1.2). Our further goal is to incorporate a fidelity function based on (1.1) directly into the denoising framework developed in [41], instead of employing (P) as a first step. To facilitate massively parallel GPU (graphics processing unit) implementation, we look for a *simple* yet fast algorithm to solve (Q), hence (P), for small m.

Various methods exists for the solution of (Q), and more generally SDLCPs. The classical methods are interior point methods roughly based around linearising the *relaxed complementarity condition* $\mathbf{X} * \mathbf{S} = \sigma\mu\mathbf{I}$ in the optimality system

$$\mathbf{MX} - \mathbf{S} = \mathbf{C}, \;\; \mathbf{X} * \mathbf{S} = \sigma\mu\mathbf{I} \quad \text{for} \quad \mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+.$$

This can be preceded by a *scaling* $\widetilde{\mathbf{X}} = \mathbf{Q}_P\mathbf{X}$ and $\underset{\sim}{\mathbf{S}} = \mathbf{Q}_P^{-1}\mathbf{S}$ to improve the conditioning and centrality of the system. Scalings popular from the linear SDP case are the XS, SX, and Nesterov-Todd scalings. Many of these methods [20, 8, 35, 24, 29] are rather involved predictor-corrector schemes, while others have expensive steps. One example of a rather simple primal-dual path-following method is presented in [21] for general monotone SDLCPs, closely related to methods for linear programming on positive definite cones (linear SDP) [23, 25, 27, 31, 28]. Employing the standard matrix product $\mathbf{X} * \mathbf{S} = \mathbf{XS}$ in the relaxed complementarity condition above, this method depends on a skew-symmetric modification of the search direction to ensure existence. Convergence is proved only in small neighbourhoods ($\gamma < 0.1$) of the central path, with complexity $O(\sqrt{m}\log\rho^{-1})$, for ρ the desired fractional duality gap.

Many more recent works [44, 45, 46, 2, 17, 14, 22] expand upon the scaling idea by using a *kernel function* or a *smoothing function* ψ [5, 30, 16, 43]. Used to derive new barrier functions for the cone $\mathcal{S}_m^+$, the effect is to convert the relaxed complementarity condition into the more general form $\Psi(\mathbf{X}, \mathbf{S}, \sigma\mu) = 0$ before linearisation. In this way new search directions can be found that give better iteration complexities than the classical scalings, generally in the class $O(\sqrt{m}\log(m\rho^{-1}))$. Notwithstanding the good iteration complexities of interior point methods, the individual steps of the methods can still become very expensive as m becomes very large: the system matrix of the Newton step has size in the order $O(m^4)$. Due to the better scalability of first-order methods to the rough solution of very large-scale problems, there has therefore recently been renewed interest in coordinate descent and alternating direction methods [48, 34, 15, 13].

In this paper, employing the symmetric product $\mathbf{X} \circ \mathbf{S} := (\mathbf{XS} + \mathbf{SX})/2$ for the relaxed complementarity condition $\mathbf{X} \circ \mathbf{S} = \sigma\mu\mathbf{I}$, we derive another simple method for (P), (Q). It is the direct analogue of the methods [23, 33, 32, 1, 9] for linear SDPs and, more generally, linear programs on general symmetric cones derived through the Jordan-algebraic [7, 19] approach. In case of quadratic SDPs, the approach of [47] is closest to ours, also based on algorithms for linear SDP. Their algorithm still employs the Nesterov-Todd scaling, attaining the iteration complexity $O(\sqrt{m}\log(m\rho^{-1}))$. Unlike in the linear SDP case, we will however see that we do not need to perform numerically expensive scaling of the primal and dual variables before linearisation of the optimality conditions to ensure $O(\sqrt{m}\kappa^2\log\rho^{-1})$ iteration complexity, for κ the condition number of $\mathbf{M}$. Our iteration complexity

will, therefore, depend on the spread of the eigenvalues of $\mathbf{M}$. It is polynomial if we assume that $\mathbf{M}$ does not, let's say, deviate too much from the identity $\mathbf{I}$, but can in the general case be considered exponential. Based on our numerical experiments in Section 5, the dependence of the complexity on κ for small m is in practise, however, very slight. For large problem instances, the main bottleneck appears to be the solution of the resulting poorly-conditioned linear system, which could possibly be avoided by scalings (that are themselves expensive). Our method and proofs could also be extended to other symmetric cones. For the sake of conciseness, we however concentrate on the positive definite cone.

In the rest of this short paper, we first study optimality conditions and search directions for (Q). We then introduce and study neighbourhoods of the central path in Section 3. We then show convergence of the proposed method in Section 4. Finally, in Section 5, we present numerical results. Further numerical application is contained in an application-oriented follow-up work [42] to [41].

2. Optimality conditions and search directions

We begin with a few definitions and known facts. We denote the symmetric (Jordan algebra) matrix product by

$$\mathbf{X} \circ \mathbf{S} := \frac{1}{2}(\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X}).$$

For later use, we also define for each $X \in \mathcal{S}_m$ the corresponding linear operator $L(\mathbf{X}) : \mathcal{S}_m \to \mathcal{S}_m$ by

$$L(\mathbf{X})\mathbf{S} := \mathbf{X} \circ \mathbf{S}.$$

We know that $L(\mathbf{X})$ is self-adjoint, $\langle L(\mathbf{X})\mathbf{V}, \mathbf{W}\rangle = \langle L(\mathbf{X})\mathbf{W}, \mathbf{V}\rangle$ for $\mathbf{V}, \mathbf{W} \in \mathcal{S}_m$. It is moreover positive definite for $\mathbf{X} \in \operatorname{int} \mathcal{S}_m^+$, where the interior

$$\operatorname{int} \mathcal{S}_m^+ = \{\mathbf{X} \in \mathcal{S}_m \mid \mathbf{X} > \mathbf{0}\}$$

is the set of positive definite elements in $\mathcal{S}_m$. Hence $L(\mathbf{X})$ is, in particular, invertible for $\mathbf{X} \in \operatorname{int} \mathcal{S}_m^+$ [33, 7, 19].

For (Q) we now easily derive the optimality conditions

$$\mathbf{M}\mathbf{X} - \mathbf{S} = \mathbf{C}, \ \mathbf{X} \circ \mathbf{S} = \mathbf{0} \quad \text{for} \quad \mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+, \tag{C}$$

where we call S the dual variable. To derive a Newton method for the solution of (Q), we first replace the *complementarity condition* $\mathbf{X} \circ \mathbf{S} = \mathbf{0}$ in (C) by the perturbed condition $\mathbf{X} \circ \mathbf{S} = \sigma\mu\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, and $\sigma, \mu > 0$ are yet to be determined. This corresponds to replacing the constraint $\mathbf{X} \geq \mathbf{0}$ by the barrier function $-\sigma\mu \log\det(\mathbf{X})$ in (P). Thus the perturbed optimality conditions become

$$\mathbf{M}\mathbf{X} - \mathbf{S} = \mathbf{C}, \ \mathbf{X} \circ \mathbf{S} = \sigma\mu\mathbf{I} \quad \text{for} \quad \mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+. \tag{C_μ}$$

Suppose then that we have $\mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+$ satisfying $\mathbf{M}\mathbf{X} - \mathbf{S} = \mathbf{C}$. We linearise ($\mathrm{C}_\mu$) at $(\mathbf{X}, \mathbf{S})$, yielding the system

$$\mathbf{M}\Delta\mathbf{X} - \Delta\mathbf{S} = 0, \quad \text{and} \tag{L1}$$

$$\mathbf{X} \circ \Delta\mathbf{S} + \mathbf{S} \circ \Delta\mathbf{X} = \sigma\mu\mathbf{I} - \mathbf{X} \circ \mathbf{S} \tag{L2}$$

for the unknowns $\Delta\mathbf{X}, \Delta\mathbf{S} \in \mathcal{S}_m$. For a yet-undetermined step-size $\alpha > 0$ we set

$$\mathbf{X}(\alpha) := \mathbf{X} + \alpha\Delta\mathbf{X}, \quad \text{and} \quad \mathbf{S}(\alpha) := \mathbf{S} + \alpha\Delta\mathbf{S}, \tag{2.1}$$

as well as

$$\mu(\alpha) := \mu(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha)), \tag{2.2}$$

where, with a slight abuse of notation, we define

$$\mu(\mathbf{V}) := \sum_j \lambda_j(\mathbf{V})/m, \quad (\mathbf{V} \in \mathcal{S}_m),$$

for $\{\lambda_i(\mathbf{V})\}_{i=1}^m$ the eigenvalues of $\mathbf{V}$. Striving for a decrease $\mu(\alpha) < \mu(0)$ we pick $\sigma \in (0,1)$ and $\mu := \mu(0) = \mu(\mathbf{X} \circ \mathbf{S})$ in (L2).

The questions now are the following: When is the system (L1), (L2) non-singular? What step-sizes α are valid to maintain in $\mathbf{X}(\alpha), \mathbf{S}(\alpha) \in \mathcal{S}_m^+$? How small can we make $\mu(\alpha)$? These bounds depend on $(\mathbf{X}, \mathbf{S})$ lying in a suitable neighbourhood of the *central path*, which are introduced and studied in the next section. First we, however, make a few remarks on linear programs and scaling.

Remark 2.1 (Linear programs with semi-definite constraints)**.** We recall that the aforementioned linear SDPs are of the form

$$\min \langle \mathbf{D}, \mathbf{X} \rangle \quad \text{such that } \mathbf{X} \in \mathcal{S}_m^+, \ \mathbf{A}\mathbf{X} = \mathbf{C}. \tag{2.3}$$

The optimality conditions may be expressed

$$\mathbf{A}\mathbf{X} = \mathbf{C}, \ \mathbf{A}^*\mathbf{y} + \mathbf{S} = \mathbf{D}, \ \mathbf{X} \circ \mathbf{S} = \mathbf{0} \quad \text{for} \quad \mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+, \ \mathbf{y} \in \mathbb{R}^K. \tag{2.4}$$

In (2.4), in contrast to (C), the primal variable $\mathbf{X}$ and the dual variable $\mathbf{S}$ are coupled *only* through the complementarity condition $\mathbf{X} \circ \mathbf{S} = \mathbf{0}$. Moreover, when (2.4) is linearised at $(\mathbf{X}, \mathbf{S}, \mathbf{y})$, to solve for the step $(\Delta\mathbf{X}, \Delta\mathbf{S}, \Delta y)$, the first two conditions become $\mathbf{A}\Delta\mathbf{X} = 0$, and $\mathbf{A}^*\Delta\mathbf{y} + \Delta\mathbf{S} = 0$. From this it follows that $\langle \Delta\mathbf{X}, \Delta\mathbf{S} \rangle = 0$. This also fails for (C), and is the crucial ingredient in the established convergence proofs for established primal-dual interior point methods for (2.3); see [33] among the other references above. Under mild conditions on the operator $\mathbf{M}$, we will, however, be able to show fast convergence of the iteration (2.1), (2.2), along similar lines as was taken in [39, 38] to study convergence properties for optimality conditions of the type

$$\mathbf{A}_1\mathbf{X} = \mathbf{C}, \ \mathbf{A}_2^*\mathbf{y} + \mathbf{S} = \mathbf{D}, \ \mathbf{X} \circ \mathbf{S} = \mathbf{0} \quad \text{for} \quad \mathbf{X}, \mathbf{S} \in \mathcal{S}_m^+, \ \mathbf{y} \in \mathbb{R}^K,$$

related to diff-convex programming.

Remark 2.2 (Scaling)**.** Choosing a scaling $0 < \mathbf{P} \in \mathcal{S}_m$, and defining the *quadratic presentation* $\mathbf{Q}_\mathbf{P} : \mathcal{S}_m \to \mathcal{S}_m$ of $\mathbf{P}$ as

$$\mathbf{Q}_\mathbf{P}(X) := \mathbf{P}\mathbf{X}\mathbf{P},$$

we may also write (C_μ) in terms of $\widetilde{\mathbf{X}} := \mathbf{Q}_\mathbf{P}\mathbf{X}$, $\underset{\sim}{\mathbf{S}} := \mathbf{Q}_\mathbf{P}^{-1}\mathbf{S}$, and $\underset{\sim}{\mathbf{M}} := \mathbf{Q}_\mathbf{P}^{-1}\mathbf{M}\mathbf{Q}_\mathbf{P}^{-1}$ as

$$\underset{\sim}{\mathbf{M}}\widetilde{\mathbf{X}} - \underset{\sim}{\mathbf{S}} = \underset{\sim}{\mathbf{C}}, \ \widetilde{\mathbf{X}} \circ \underset{\sim}{\mathbf{S}} = \sigma\mu\mathbf{I} \quad \text{for} \quad \widetilde{\mathbf{X}}, \underset{\sim}{\mathbf{S}} \in \mathcal{S}_m^+. \tag{$\widetilde{\mathrm{C}}_\mu$}$$

In many primal-dual interior point methods for linear SDPs [23, 33], this type of scaling is typically performed at each step to force fast convergence. The idea is to choose $\mathbf{P}$ such that $\mathbf{X}$ and $\mathbf{S}$ operator-commute, $L(\mathbf{X})L(\mathbf{S}) = L(\mathbf{S})L(\mathbf{X})$. For the XS-method $\mathbf{P} = \mathbf{X}^{1/2}$, for the SX-method $\mathbf{P} = \mathbf{S}^{1/2}$, and for the Nesterov-Todd method $\mathbf{P} = (\mathbf{Q}_{\mathbf{X}^{1/2}}(\mathbf{Q}_{\mathbf{X}^{1/2}}\mathbf{S})^{-1/2})^{-1/2}$ [33]. We will not need this type of computationally expensive (matrix square root!) scalings for fast convergence. However our bounds will depend on $\mathbf{M}$. If we did perform scaling, we could obtain bounds that do not depend on $\mathbf{M}$, following the proof of [33].

Remark 2.3 (An easy special case)**.** Suppose that $\mathbf{M}\mathbf{X} = \mathbf{Q}_\mathbf{A}(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}$ for some $\mathbf{A} \in \operatorname{int}\mathcal{S}_m^+$. Then $\mathbf{M}\mathcal{S}_m^+ = \mathcal{S}_m^+$ [33, 7, 19]. In the previous remark, let us choose $\mathbf{P} = \mathbf{A}^{1/2}$. The optimality conditions ($\widetilde{\mathrm{C}}_\mu$) for $\mu = 0$ then become

$$\widetilde{\mathbf{X}} - \underset{\sim}{\mathbf{S}} = \underset{\sim}{\mathbf{C}}, \ \widetilde{\mathbf{X}} \circ \underset{\sim}{\mathbf{S}} = \mathbf{0} \quad \text{for} \quad \widetilde{\mathbf{X}}, \underset{\sim}{\mathbf{S}} \in \mathcal{S}_m^+. \tag{2.5}$$

The solution $\widetilde{\mathbf{X}}$ of (2.5) is simply the Frobenius-norm projection of $\underset{\sim}{\mathbf{C}}$ to $\mathcal{S}_m^+$, and can easily be solved by projection of the eigenvalues $(\lambda_1(\underset{\sim}{\mathbf{C}}), \ldots, \lambda_m(\underset{\sim}{\mathbf{C}}))$ to $[0, \infty)^m$. Thus the solution $\mathbf{X} = \mathbf{Q}_{\mathbf{A}^{-1/2}}(\widetilde{\mathbf{X}})$ of (C) can also be easily calculated in this special case.

3. Neighbourhoods of the central path

Let now $\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V} := \mathbf{V} - \mathbf{I}\mu(\mathbf{V})$ denote the projection of $\mathbf{V} \in \mathcal{S}_m$ to the subspace orthogonal to the identity $\mathbf{I}$. The spectrum of $\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V}$ is then $\{\lambda_i(\mathbf{V}) - \mu(\mathbf{V})\}_{i=1}^m$.

With this, we now let $\gamma \in (0,1)$, and define

$$\mathcal{N}_{\bullet}^{*}(\gamma) := \{(\mathbf{X},\mathbf{S}) \in \operatorname{int}\mathcal{S}_m^{+} \times \operatorname{int}\mathcal{S}_m^{+} \mid \|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}\circ\mathbf{S})\|_{\bullet} \le \gamma\mu(\mathbf{X}\circ\mathbf{S})\} \tag{3.1}$$

for $\bullet \in \{F, 2, -\infty\}$. These correspond to the short-step, semi-long-step, and long-step neighbourhoods of $\mathcal{S}_m^{+} \times \mathcal{S}_m^{+}$, and are obtained, respectively, with the Frobenius norm

$$\|\mathbf{V}\|_F := \sqrt{\textstyle\sum_{i=1}^m \lambda_i(\mathbf{V})^2} = \sqrt{\langle \mathbf{V}, \mathbf{V}\rangle},$$

the operator 2-norm

$$\|\mathbf{V}\|_2 := \max_{i=1,\dots,m} |\lambda_i(\mathbf{V})| = \max_{\mathbf{x}\in\mathbb{R}^m} \|\mathbf{V}\mathbf{x}\|_2/\|\mathbf{x}\|_2,$$

and, abusing norm notation for the sake of convenience, the function

$$\|\mathbf{V}\|_{-\infty} := -\min_i \lambda_i(\mathbf{V}).$$

For $\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V}$ these have expressions

$$\begin{aligned}
\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V}\|_F &= \sqrt{\textstyle\sum_i (\lambda_i(\mathbf{V}) - \mu(\mathbf{V}))^2},\\
\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V}\|_2 &= \max_i |\lambda_i(\mathbf{V}) - \mu(\mathbf{V})|, \quad \text{and}\\
\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{V}\|_{-\infty} &= \mu(\mathbf{V}) - \min \lambda_i(\mathbf{V}).
\end{aligned} \tag{3.2}$$

It easily follows that

$$\mathcal{N}_F^{*}(\gamma) \subset \mathcal{N}_2^{*}(\gamma) \subset \mathcal{N}_{-\infty}^{*}(\gamma).$$

The following proposition is one of the crucial ingredients for our convergence proof.

Proposition 3.1. *Suppose* $(\mathbf{X},\mathbf{S}) \in \mathcal{N}_{-\infty}^{*}(\gamma)$. *Then*

$$2(1-\gamma)\mu(\mathbf{X}\circ\mathbf{S}) \le \Big(\min_{i=1,\dots,m} \lambda_i(\mathbf{X}+\mathbf{S})\Big)^2. \tag{3.3}$$

Proof. We get from (3.2), (3.1) that

$$(1-\gamma)\mu(\mathbf{X}\circ\mathbf{S}) \le \min_i \lambda_i(\mathbf{X}\circ\mathbf{S}). \tag{3.4}$$

This shows that $\mathbf{X}\circ\mathbf{S} \in \mathcal{S}_m^{+}$, wherefore also

$$\langle \mathbf{X}\mathbf{y}, \mathbf{S}\mathbf{y}\rangle = \langle \mathbf{y}, (\mathbf{X}\circ\mathbf{S})\mathbf{y}\rangle > 0, \quad (\mathbf{y}\in\mathbb{R}^m).$$

Therefore

$$\begin{aligned}
\min_i \lambda_i(\mathbf{X}\circ\mathbf{S}) &= \min_{\|\mathbf{y}\|=1} \langle \mathbf{y}, (\mathbf{X}\circ\mathbf{S})\mathbf{y}\rangle\\
&\le \min_{\|\mathbf{y}\|=1} \frac{1}{2}\big(\|\mathbf{X}\mathbf{y}\|_2^2 + \|\mathbf{S}\mathbf{y}\|_2^2\big) + \langle \mathbf{X}\mathbf{y}, \mathbf{S}\mathbf{y}\rangle\\
&= \min_{\|\mathbf{y}\|=1} \frac{1}{2}\big(\|\mathbf{X}\mathbf{y} + \mathbf{S}\mathbf{y}\|_2^2\big)\\
&= \frac{1}{2}\Big(\min_i \lambda_i(\mathbf{X}+\mathbf{S})\Big)^2.
\end{aligned}$$

In the final step we have used the fact that $\mathbf{X}+\mathbf{S} \succeq \mathbf{0}$. Recalling (3.4), the claim (3.3) follows. □

4. The method and its convergence

We now begin to study rates of convergence for the proposed method, consisting of the updates (2.1), (2.2). We assume that the linear operator $\mathbf{M} : \mathcal{S}_m \to \mathcal{S}_m$ is self-adjoint and satisfies for some $0 < \theta \le \Theta < \infty$ the condition

$$\Theta\langle \mathbf{V}, \mathbf{V}\rangle \ge \langle \mathbf{V}, \mathbf{M}\mathbf{V}\rangle \ge \theta\langle \mathbf{V}, \mathbf{V}\rangle, \quad (\mathbf{V} \in \mathcal{S}_m). \tag{A-M}$$

We begin by computing bounds on α for staying within $\mathcal{N}_\bullet^*(\gamma)$. With the notation

$$L := \{(\mathbf{X}, \mathbf{S}) \in \mathcal{S}_m^+ \times \mathcal{S}_m^+ \mid \mathbf{M}\mathbf{X} - \mathbf{S} = \mathbf{C}\}$$

for the feasible set, we have the following lemma.

Lemma 4.1. *If* $(\mathbf{X}, \mathbf{S}) \in \mathcal{N}_\bullet^*(\gamma) \cap L$ *for some* $\bullet \in \{F, 2, -\infty\}$, *then* $(\mathbf{X}(\alpha), \mathbf{S}(\alpha)) \in \mathcal{N}_\bullet^*(\gamma) \cap L$ *for* $\alpha \in [0, \bar\alpha]$, *where*

$$\bar\alpha := \min\{1, \sigma\gamma\mu/(2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F)\}. \tag{4.1}$$

Proof. This proof follows the outline of the proof of [33, Lemma 32], slightly modifying the proof to accommodate for the fact that $\langle \Delta\mathbf{X}, \Delta\mathbf{S}\rangle = 0$ does not hold; see also [39, 38].

Clearly $(\mathbf{X}, \mathbf{S}) \in L$ and (L1) imply $(\mathbf{X}(\alpha), \mathbf{S}(\alpha)) \in L$. For the remainder, it will suffice to prove that

$$\|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha))\|_\bullet < \gamma\mu(\alpha) \quad \text{for} \quad \alpha \in (0, \bar\alpha]. \tag{4.2}$$

Indeed, irrespective of the choice of $\bullet$, (4.2) then holds for $\bullet = -\infty$ as well. Consequently

$$(1-\gamma)\mu(\alpha) < \min_i \lambda_i(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha)) = \min_i \lambda_i(\mathbf{X}(\alpha)\mathbf{S}(\alpha)),$$

where the last equality holds because $\mathbf{S}\mathbf{X}$ and $\mathbf{X}\mathbf{S}$ have the same spectrum for any $\mathbf{X}, \mathbf{S} \in \mathcal{S}_m$. But then, taking the power of m on both sides, we get

$$((1-\gamma)\mu(\alpha))^m < \det(\mathbf{X}(\alpha)\mathbf{S}(\alpha)) = \det(\mathbf{X}(\alpha))\det(\mathbf{S}(\alpha)).$$

Now, by the continuity of the involved quantities in α, this condition would be violated if at some point $\alpha \in (0, \bar\alpha]$ one of the determinants were zero, that is, either $\mathbf{X}(\alpha)$ or $\mathbf{S}(\alpha)$ reached the boundary $\partial\mathcal{S}_m^+$. Thus (4.2) implies that $\mathbf{X}(\alpha), \mathbf{S}(\alpha) \in \operatorname{int}\mathcal{S}_m^+$ for $\alpha \in (0, \bar\alpha]$. Consequently, still by (4.2), $(\mathbf{X}(\alpha), \mathbf{S}(\alpha)) \in \mathcal{N}_\bullet^*(\gamma)$ for every $\alpha \in [0, \bar\alpha]$, as claimed.

It remains to prove (4.2). With the notation $\mathbf{Z} := \Delta\mathbf{X} \circ \Delta\mathbf{S}$, we have

$$\begin{aligned}\mu(\alpha) &= \mu(\mathbf{X} \circ \mathbf{S}) + \alpha\mu(\mathbf{X} \circ \Delta\mathbf{S} + \mathbf{S} \circ \Delta\mathbf{X}) + \alpha^2\mu(\Delta\mathbf{X} \circ \Delta\mathbf{S}) \\ &= \mu + \alpha(\sigma - 1)\mu + \alpha^2\mu(\mathbf{Z}) \\ &= (1-\alpha)\mu + \alpha\sigma\mu + \alpha^2\mu(\mathbf{Z}),\end{aligned} \tag{4.3}$$

as well as

$$\begin{aligned}\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha)) &= \mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X} \circ \mathbf{S}) + \alpha\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X} \circ \Delta\mathbf{S} + \mathbf{S} \circ \Delta\mathbf{X}) + \alpha^2\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z} \\ &= \mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X} \circ \mathbf{S}) + \alpha\mathbf{P}_{\mathbf{I}}^{\perp}(\sigma\mu\mathbf{I} - \mathbf{X} \circ \mathbf{S}) + \alpha^2\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z} \\ &= (1-\alpha)\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X} \circ \mathbf{S}) + \alpha^2\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z}.\end{aligned} \tag{4.4}$$

To approximate the norm $\|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha))\|_\bullet$, for $\bullet = F$ we can use the triangle inequality on (4.4), whereas for $\bullet = 2, -\infty$, we apply, respectively, the inequalities

$$\begin{aligned}\max \lambda_i(\mathbf{V} + \mathbf{W}) &\le \max \lambda_i(\mathbf{V}) + \|\mathbf{W}\|_F, \quad \text{and} \\ -\min \lambda_i(\mathbf{V} + \mathbf{W}) &\le -\min \lambda_i(\mathbf{V}) + \|\mathbf{W}\|_F, \quad (\mathbf{V}, \mathbf{W} \in \mathcal{S}_m).\end{aligned}$$

Therefore, for all $\bullet \in \{F, 2, -\infty\}$, we have the approximation

$$\begin{aligned}\|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}(\alpha) \circ \mathbf{S}(\alpha))\|_{\bullet} &\leq (1-\alpha)\|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X} \circ \mathbf{S})\|_{\bullet} + \alpha^2\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z}\|_F \\ &\leq (1-\alpha)\gamma\mu + \alpha^2\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z}\|_F.\end{aligned}$$

Comparing this approximation against the expansion (4.3) of $\mu(\alpha)$, we find that (4.2) holds if

$$\alpha^2\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z}\|_F < (1-\alpha-|1-\alpha|+\alpha\sigma)\gamma\mu + \gamma\alpha^2\mu(\mathbf{Z}).$$

Minding that $\mu(\mathbf{Z}) > 0$ by (L1), and that $\|\mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{Z}\|_F \leq 2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F$, this follows if

$$2\alpha^2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F \leq (1-\alpha-|1-\alpha|+\alpha\sigma)\gamma\mu.$$

The latter clearly holds for $\alpha \in [0, \bar{\alpha}] \subset [0, 1]$. This completes the proof of (4.2) and the lemma. □

The following lemma provides the estimate on the fractional decrease $\mu(\bar{\alpha})/\mu$.

Lemma 4.2. *Assume the conditions of Lemma 4.1, and suppose $0 < \sigma < m/(m+\gamma)$. Then*

$$\begin{aligned}\delta := 1 - \mu(\bar{\alpha})/\mu &\geq (1-\sigma)\bar{\alpha}/2 \\ &= (1-\sigma)/2 \cdot \min\{1, \sigma\gamma\mu/(2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F)\}.\end{aligned} \tag{4.5}$$

Proof. Observe that, thanks to the condition $\sigma < m/(m+\gamma)$, we have

$$\begin{aligned}(1-\sigma)m\mu/(2\langle\Delta\mathbf{X}, \Delta\mathbf{S}\rangle) &\geq \sigma\gamma\mu/(2\langle\Delta\mathbf{X}, \Delta\mathbf{S}\rangle) \\ &\geq \sigma\gamma\mu/(2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F) \geq \bar{\alpha}.\end{aligned}$$

Therefore, using the expansion (4.3), we may calculate

$$1 - \mu(\bar{\alpha})/\mu = (1-\sigma)\bar{\alpha} - \bar{\alpha}^2\langle\Delta\mathbf{X}, \Delta\mathbf{S}\rangle/(m\mu) \geq (1-\sigma)\bar{\alpha}/2. \tag{4.6}$$

Inserting (4.1) into (4.6) gives (4.5). □

Given a lower bound $\hat{\delta} \leq \delta$, a standard argument (see, e.g., [28]) shows that $\hat{\delta}^{-1}\log\rho^{-1}$ steps are sufficient to ensure that $\mu \leq \rho\overline{\mu}$ for an initial $\overline{\mu} > 0$ and desired decrease factor $\rho \in (0, 1)$. To obtain the lower bound $\hat{\delta}$, and hence fast decrease in μ, by the previous lemma it suffices to bound $\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F/\mu$ from above. The standard proofs (see, e.g., [33]) for the semi-definite programming problem (2.3) rely at this point on the commutativity of $L(\widetilde{\mathbf{X}})$ and $L(\underset{\sim}{\mathbf{S}})$, where $\widetilde{\mathbf{X}}$ and $\underset{\sim}{\mathbf{S}}$ are the scaled variables as in Remark 2.2. We do not do so, as we want to avoid the computationally expensive scalings (involving matrix square roots), and show that we can avoid them, as do many other methods for quadratic SDPs and SDLCPs, referenced in the introduction.

Proposition 4.1 below is our most crucial ingredient for the convergence proof, and the main divergence from the proofs in [33] for linear SDPs. We show bounds for the short-step neighbourhood $\mathcal{N}_F^*(\gamma)$ and the semi-long-step neighbourhood $\mathcal{N}_2^*(\gamma)$.

Proposition 4.1. *Suppose* $\mathbf{M}$ *satisfies* (A-M), *and* $(\mathbf{X}, \mathbf{S}) \in \mathcal{N}_F^*(\gamma)$. *Then*

$$\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F/\mu \leq \frac{((1-\sigma)\sqrt{m}+\gamma)^2}{2(1-\gamma)}(\Theta/\theta)^2. \tag{4.7}$$

If $(\mathbf{X}, \mathbf{S}) \in \mathcal{N}_2^*(\gamma)$, *then*

$$\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F/\mu \leq \frac{(1-\sigma+\gamma)^2}{2(1-\gamma)}m(\Theta/\theta)^2. \tag{4.8}$$

Moreover, in both cases, there exists a unique solution $(\Delta\mathbf{X}, \Delta\mathbf{S})$ *to the system* (L1), (L2).

Proof. Let us set

$$\mathbf{A} := L(\mathbf{S}) + L(\mathbf{X})\mathbf{M}.$$

Assuming that $\mathbf{A}$ is invertible, from (L1), (L2), we have

$$\Delta\mathbf{X} = \mathbf{A}^{-1}(\sigma\mu\mathbf{I} - \mathbf{X}\circ\mathbf{S}), \quad \text{and} \quad \Delta\mathbf{S} = \mathbf{M}\Delta\mathbf{X}. \tag{4.9}$$

Hence

$$\frac{\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F}{\mu} \le \Theta\frac{\|\Delta\mathbf{X}\|_F^2}{\mu} \le \Theta\frac{|\mathbf{A}^{-1}|_{\max}^2\|\sigma\mu\mathbf{I} - \mathbf{X}\circ\mathbf{S}\|_F^2}{\mu}, \tag{4.10}$$

where we denote

$$|\mathbf{A}^{-1}|_{\max} := \max\{\|\mathbf{A}^{-1}\mathbf{V}\|_F \mid \mathbf{V}\in\mathcal{S}_m,\ \|\mathbf{V}\|_F = 1\} \le 1/|\mathbf{A}|_{\min}$$

and

$$|\mathbf{A}|_{\min} := \min\{\|\mathbf{A}\mathbf{V}\|_F \mid \mathbf{V}\in\mathcal{S}_m,\ \|\mathbf{V}\|_F = 1\}.$$

We want to bound $|\mathbf{A}|_{\min}$ from below. Picking $\mathbf{V}\in\mathcal{S}_m$ with $\|\mathbf{V}\|_F = 1$, we calculate

$$\begin{aligned}\|\mathbf{A}\mathbf{V}\|_F &= \|L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{S}+\Theta\mathbf{X})^{-1}\mathbf{A}\mathbf{V}\|_F\\ &\ge |L(\mathbf{S}+\Theta\mathbf{X})|_{\min}\|L(\mathbf{S}+\Theta\mathbf{X})^{-1}\mathbf{A}\mathbf{V}\|_F,\end{aligned} \tag{4.11}$$

We know that (see, e.g., [33])

$$|L(\mathbf{S}+\Theta\mathbf{X})|_{\min} = \min_i \lambda_i(\mathbf{S}+\Theta\mathbf{X}).$$

Observing that $(\mathbf{X},\mathbf{S})\in\mathcal{N}^*_{-\infty}(\gamma)$ implies $(\Theta\mathbf{X},\mathbf{S})\in\mathcal{N}^*_{-\infty}(\gamma)$, and referring to Proposition 3.1, we thus find that

$$|L(\mathbf{S}+\Theta\mathbf{X})|_{\min} \ge \sqrt{2(1-\gamma)\Theta\mu}. \tag{4.12}$$

To estimate $\|L(\mathbf{S}+\Theta\mathbf{X})^{-1}\mathbf{A}\mathbf{V}\|_F$, we observe that

$$L(\mathbf{S}+\Theta\mathbf{X})^{-1}\mathbf{A} = \mathbf{I} + L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})(\mathbf{M}-\Theta\mathbf{I}).$$

By Lemma 4.3 below, we have

$$|L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})|_{\max} \le 1/\Theta.$$

Hence, employing (A-M) to estimate

$$|\mathbf{M}-\Theta\mathbf{I}|_{\max} \le \Theta - \theta,$$

we obtain the bound

$$\begin{aligned}\|L(\mathbf{S}+\Theta\mathbf{X})^{-1}\mathbf{A}\mathbf{V}\|_F &\ge 1 - \|L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})(\mathbf{M}-\Theta\mathbf{I})\mathbf{V}\|_F.\\ &\ge 1 - |L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})|_{\max}|\mathbf{M}-\Theta\mathbf{I}|_{\max}\\ &\ge 1 - (1/\Theta)(\Theta-\theta)\\ &= \theta/\Theta.\end{aligned} \tag{4.13}$$

Since $\mathbf{V}\in J$, $\|\mathbf{V}\|_F = 1$ was arbitrary, applying (4.12) and (4.13) in (4.11) yields

$$|\mathbf{A}|_{\min} \ge \|\mathbf{A}\mathbf{V}\|_F \ge \sqrt{2(1-\gamma)\theta^2\mu/\Theta} > 0. \tag{4.14}$$

As this bound shows that $\mathbf{A}$ is invertible, the system (L1), (L2) has the unique solution (4.9). Moreover, applying (4.14) in (4.10) yields

$$\frac{\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F}{\mu} \le \frac{\|\sigma\mu\mathbf{I} - \mathbf{X}\circ\mathbf{S}\|_F^2}{2(1-\gamma)\mu^2}(\Theta/\theta)^2. \tag{4.15}$$

In the case $(\mathbf{X},\mathbf{S}) \in \mathcal{N}_F^*(\gamma)$, using the fact that $\|\mathbf{X}\circ\mathbf{S}-\mu\mathbf{I}\|_F \le \gamma\mu$, and the triangle inequality, we have

$$\begin{aligned}\|\sigma\mu\mathbf{I}-\mathbf{X}\circ\mathbf{S}\|_F &\le \|\sigma\mu\mathbf{I}-\mu\mathbf{I}\|_F + \|\mathbf{X}\circ\mathbf{S}-\mu\mathbf{I}\|_F \\ &\le ((1-\sigma)\sqrt{m}+\gamma)\mu.\end{aligned} \tag{4.16}$$

Hence, (4.7) follows from (4.15).

In the case $(\mathbf{X},\mathbf{S}) \in \mathcal{N}_2^*(\gamma)$, using $\|\mathbf{X}\circ\mathbf{S}-\mu\mathbf{I}\|_2 \le \gamma\mu$, we have

$$\|\mu\mathbf{I}-\mathbf{X}\circ\mathbf{S}\|_F \le \sqrt{m}\|\mu\mathbf{I}-\mathbf{X}\circ\mathbf{S}\|_2 \le \gamma\mu\sqrt{m},$$

so that, similarly to (4.16), we obtain

$$\|\sigma\mu\mathbf{I}-\mathbf{X}\circ\mathbf{S}\|_F \le (1-\sigma+\gamma)\mu\sqrt{m}.$$

Hence (4.15) shows (4.8), concluding the proof. □

We needed the following lemma to prove Proposition 4.1.

Lemma 4.3. *Suppose* $\mathbf{X},\mathbf{S} \in \operatorname{int}\mathcal{S}_m^+$, *and let* $\Theta > 0$. *Then*

$$|L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})|_{\max} < 1/\Theta.$$

Proof. We have to prove that

$$\|L(\mathbf{S}+\Theta\mathbf{X})^{-1}L(\mathbf{X})\mathbf{V}\|_F < \|\mathbf{V}\|_F/\Theta, \quad (\mathbf{V}\in\mathcal{S}_m).$$

Squaring, expanding, and reorganising, this is written

$$\Theta^2\langle L(\mathbf{X})L(\mathbf{S}+\Theta\mathbf{X})^{-2}L(\mathbf{X})\mathbf{V},\mathbf{V}\rangle < \langle\mathbf{V},\mathbf{V}\rangle, \quad (\mathbf{V}\in\mathcal{S}_m).$$

Writing $\mathbf{V} = L(\mathbf{X})^{-1}L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})\mathbf{W}$, we get the equivalent condition

$$\Theta^2\langle L(\mathbf{X})^2\mathbf{W},\mathbf{W}\rangle < \langle L(\mathbf{X})L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})^{-2}L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})\mathbf{W},\mathbf{W}\rangle \tag{4.17}$$

for all $\mathbf{W}\in\mathcal{S}_m$. Expanding, we have

$$\begin{aligned}&L(\mathbf{X})L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})^{-2}L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X}) \\ &\quad= L(\mathbf{X})L(\mathbf{S})L(\mathbf{X})^{-2}L(\mathbf{S})L(\mathbf{X}) \\ &\qquad+ \Theta(L(\mathbf{X})L(\mathbf{S})+L(\mathbf{S})L(\mathbf{X})) \\ &\qquad+ \Theta^2L(\mathbf{X})^2.\end{aligned} \tag{4.18}$$

We know from the proof of [9, Corollary 4.4] that

$$\langle\big(L(\mathbf{X})L(\mathbf{S})+L(\mathbf{S})L(\mathbf{X})\big)\mathbf{W},\mathbf{W}\rangle > \langle L(\mathbf{S}\circ\mathbf{X})\mathbf{W},\mathbf{W}\rangle > 0$$

under the condition $\mathbf{X},\mathbf{S}\in\operatorname{int}\mathcal{S}_m^+$. Moreover, clearly by symmetricity

$$\langle L(\mathbf{X})L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})^{-2}L(\mathbf{S}+\Theta\mathbf{X})L(\mathbf{X})\mathbf{W},\mathbf{W}\rangle > 0.$$

Therefore, the expansion (4.18) shows (4.17), concluding the proof. □

We now concentrate on convergence in the case $(\mathbf{X},\mathbf{S})\in\mathcal{N}_F^*(\gamma)$. Recalling from (4.5) that

$$\delta \ge (1-\sigma)/2\cdot\min\{1,\sigma\gamma\mu/(2\|\Delta\mathbf{X}\|_F\|\Delta\mathbf{S}\|_F)\},$$

provided that $0 < \sigma \leq m/(m+\gamma)$, we now find with (4.7) that

$$\delta \geq (1-\sigma)/2 \cdot \min\left\{1, \frac{\sigma\gamma(1-\gamma)}{((1-\sigma)\sqrt{m}+\gamma)^2}(\theta/\Theta)^2\right\}. \tag{4.19}$$

If we pick $\sigma = 1-\gamma/\sqrt{m}$, which can be seen to satisfy $1-\gamma \leq \sigma \leq m/(m+\gamma)$, we obtain

$$\begin{aligned}\delta &\geq \gamma/(2\sqrt{m}) \cdot \min\left\{1, \frac{(1-\gamma/\sqrt{m})(1-\gamma)}{4\gamma}(\theta/\Theta)^2\right\} \\ &\geq m^{-1/2}\min\left\{\frac{\gamma}{2}, \frac{(1-\gamma)^2}{8}(\theta/\Theta)^2\right\}.\end{aligned}$$

If $\gamma = 1/2$, we in particular get

$$\delta \geq m^{-1/2}(\theta/\Theta)^2/32.$$

The results of this section are summarised in the following algorithm description and theorem.

Algorithm 4.1. The proposed interior point method is as follows.

1. Choose target accuracy $\underline{\mu} > 0$, and parameters $\gamma \in (0,1)$, $\sigma \in (0, m/(m+\gamma)]$. Pick an initial iterate $(\mathbf{X}^0, \mathbf{S}^0) \in \mathcal{N}_\bullet(\gamma) \cap L$ for some choice of neighbourhood $\bullet \in \{F, 2\}$. Calculate $\mu^0 = \langle \mathbf{X}^0, \mathbf{S}^0\rangle/m$, and set $i := 0$.
2. Solve $\Delta\mathbf{X}^i$ from
$$\big(L(\mathbf{S}^i) + L(\mathbf{X}^i)\mathbf{M}\big)\Delta\mathbf{X}^i = \sigma\mu^i\mathbf{I} - \mathbf{X}^i \circ \mathbf{S}^i, \tag{4.20}$$
and set
$$\Delta\mathbf{S}^i = \mathbf{M}\Delta\mathbf{X}^i.$$
3. Calculate the step length
$$\bar{\alpha} := \min\{1, \sigma\gamma\mu/(2\|\Delta\mathbf{X}^i\|_F\|\Delta\mathbf{S}^i\|_F)\},$$
and update $\mathbf{X}^{i+1} := \mathbf{X}^i + \bar{\alpha}\Delta\mathbf{X}^i$, and $\mathbf{S}^{i+1} := \mathbf{S}^i + \bar{\alpha}\Delta\mathbf{S}^i$, as well as $\mu^{i+1} = \langle \mathbf{X}^{i+1}, \mathbf{S}^{i+1}\rangle/m$,
4. If $\mu^{i+1} < \underline{\mu}$, terminate. Otherwise continue from Step 2 with $i := i+1$.

We now state the convergence proof for the short-step neighbourhood $\mathcal{N}_F(\gamma)$.

Theorem 4.1. *Suppose (A-M) holds. Let $(\mathbf{X}^0, \mathbf{S}^0) \in \mathcal{N}_F(\gamma) \cap L$ and choose target accuracy $\underline{\mu}$. Choose $\sigma = 1-\gamma/\sqrt{m}$. Then in Algorithm 4.1, we have*

$$\mu^i < \underline{\mu} \quad \textit{for} \quad i > \delta^{-1}\log(\mu_0/\underline{\mu}),$$

where the constant

$$\delta = m^{-1/2}\min\left\{\frac{\gamma}{2}, \frac{(1-\gamma)^2}{8}(\theta/\Theta)^2\right\}. \tag{4.21}$$

In particular, if $\gamma = 1/2$, then

$$\mu^i < \underline{\mu} \quad \textit{for} \quad i > 32\sqrt{m}(\Theta/\theta)^2\log(\mu_0/\underline{\mu}).$$

Proof. Follows from the discussion above with $\rho = \underline{\mu}/\mu_0$. □

Remark 4.1. For the semi-long-step neighbourhood $\mathcal{N}_2(\gamma)$ we get for any $\gamma \in (0,1)$ and $\sigma \in (0, m/(m+\gamma))$ by application of (4.5), (4.8) the bound $i > \zeta m(\Theta/\theta)^2\log(\mu_0/\underline{\mu})$ for some constant $\zeta = \zeta(\gamma, \sigma)$.

Remark 4.2. In practice, we may perform initialisation as follows. We pick $\mathbf{X}^0 = \beta I$ for a yet unknown $\beta > 0$. Then $\mathbf{S}^0 = \beta\mathbf{MI} - \mathbf{C}$. The condition $(\mathbf{X}^0, \mathbf{S}^0) \in \mathcal{N}_\bullet(\gamma)$, namely

$$\|\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{X}^0 \circ \mathbf{S}^0)\|_F \leq \gamma\mu(\mathbf{X}^0 \circ \mathbf{S}^0)$$

gives after division by β the condition

$$\|\beta\mathbf{P}_{\mathbf{I}}^{\perp}(\mathbf{MI}) - \mathbf{P}_{\mathbf{I}}^{\perp}\mathbf{C}\|_F \leq \beta\mu(\mathbf{MI}) - \mu(\mathbf{C}).$$

Squaring both sides, we get a second-order polynomial equation on β, from which we get a lower bound β_1 on β. Another lower bound β_2 on β is given by the condition $\mathbf{S}^0 \geq 0$. We may then choose $\beta := \max\{\beta_1, \beta_2, 0\}$.

Remark 4.3. An alternative initialisation strategy is to solve $\bar{\mathbf{X}}^0$ from $\mathbf{M}\bar{\mathbf{X}}^0 = \mathbf{C}$. Then we check if already $\bar{\mathbf{X}}^0 \geq \mathbf{0}$ (by Sylvester's criterion), in which case we may skip Algorithm 4.1. This is crucial for efficiency in our intended DTI denoising application. If $\bar{\mathbf{X}}^0 \not\geq \mathbf{0}$, we set $\mathbf{X}^0 := \bar{\mathbf{X}}^0 + \beta\mathbf{I}$ for an unknown $\beta > 0$, and $\mathbf{S}^0 := \mathbf{M}\mathbf{X}^0 - \mathbf{C}$. Then we calculate lower bounds for β analogously to Remark 4.2.

5. Numerical results

Scalability ($\mathbf{M} = \mathbf{I}$) We tested the actual performance of the proposed method on the problem (Q) numerically. As a first test case we took the identity operator $\mathbf{M} = \mathbf{I}$ (in which case we could just use the QR algorithm to perform projection! – recall Remark 2.3) for varying dimensions m with the parameters $\gamma = 0.5$ and $\sigma = 1 - \gamma/\sqrt{m}$. The results of these experiments, for a decrease $\rho = 0.001$ of μ are plotted in Figure 1a, along with the iteration complexity upper bound from Theorem 4.1. The reported iteration count is the median over 10 samples of $\mathbf{C} \in \mathcal{S}_m$, with each component drawn independently from the uniform distribution on $[-1, 1]$. The maximum and minimum in the test run have a difference of at most ± 1 to the median. As we see, the method performs better than the prediction. However, as m becomes large, solving the system (4.20) (by LU decomposition) at each step starts to become prohibitively expensive, so other approaches are needed, such as the inexact steps of [35]. An improved solver or preconditioning of the linear system could also help. A level of preconditioning is conventionally achieved through the use scalings. They could be used with our method, but are themselves expensive.

Parameter choice ($m = 3$) We are mostly interested in the case $m = 3$. For this we report the iteration counts for varying parameters γ and σ in Figure 1b. Defining the symmetric presentation $\mathbf{PA}$ of

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,2} & a_{2,2} & a_{2,3} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{bmatrix} \in \mathcal{S}_3$$

by

$$\mathbf{PA} = (a_{1,1}, a_{1,2}, a_{2,2}, a_{1,3}, a_{2,3}, a_{3,3}),$$

this time we use the fixed operator

$$\mathbf{M} = \mathbf{B}^*\mathbf{B} \approx \mathbf{P}^* \begin{bmatrix} 9.3716 & -0.0146 & 3.3252 & 0.0064 & -0.0062 & 3.2670 \\ -0.0146 & 3.3252 & 0.0107 & -0.0062 & -0.0028 & -0.0008 \\ 3.3252 & 0.0107 & 9.5023 & -0.0028 & 0.0565 & 3.2863 \\ 0.0064 & -0.0062 & -0.0028 & 3.2670 & -0.0008 & -0.0062 \\ -0.0062 & -0.0028 & 0.0565 & -0.0008 & 3.2863 & -0.0711 \\ 3.2670 & -0.0008 & 3.2863 & -0.0062 & -0.0711 & 9.3691 \end{bmatrix} \mathbf{P}$$

with $\Theta/\theta \approx 4.9$ and $\mathbf{B} : \mathcal{S}_3 \to \mathbb{R}^{52}$, constructed from a real DTI measurement setup. The reported data points are again the median over 10 samples of $\mathbf{c} \in \mathbb{R}^{52}$, with each component drawn independently uniformly from $[-1, 1]$. The difference of the maximum and minimum iteration counts to the median is still too small to be observable in the figure. Apparently $\gamma = 0.9$ and $\sigma = 1 - \gamma/\sqrt{m} \approx 0.48$ would be a good choice of parameters, with a very small iteration count of about 11. This is significantly smaller than the theoretical bound of 22982 given by Theorem 4.1. The theorem provides a much better bound of 140 for $\gamma = 3 - 2\sqrt{2} \approx 0.17$ and $\sigma = 1 - \gamma/\sqrt{m} \approx 0.9$.

Alternative methods We also tested our method against various existing approaches, for which codes are publicly available. Besides solvers for the quadratic problem, we tested solvers for the SOCP reformulation of (P), namely

$$\min_{\mathbf{X}\geq\mathcal{S}_m^+,\,\|\mathbf{v}\|_2\leq t} t \quad \text{such that} \quad \mathbf{B}\mathbf{X}-\mathbf{c}=\mathbf{v}.$$

The computations for this reformulation were done using the CVX front-end [12, 11] to SDPT3 [36, 37] and the commercial solver Mosek. To get an overview of the overhead of using CVX, we also performed the computations with SDPT3 directly. With Mosek this does not make sense for small problems, as an external program has to be called from Matlab, involving significant overhead. We also used QSDP [35] to solve the quadratic problem (P) directly. Again, for a number of dimensions m, we generated 10 random operators $\mathbf{B}\ \mathbf{B}:\mathcal{S}_m\to\mathcal{S}_m$ and matrices $\mathbf{C}\in\mathcal{S}_m$. The latter was drawn as before, while the former was generated in in such a manner that the condition numbers of $\mathbf{M}$ do not become unreasonable, which causes very slow convergence from our proposed method for large m. Namely, we took each component of $\mathbf{B}_0$ independently uniformly from $[0,1]$, and then set $\mathbf{B}=\sqrt{\mathbf{B}^*\mathbf{B}/k^2+I/k}$, for $k=(m+1)m/2$ the number of independent components of $\mathbf{B}_0$. This resulted in condition numbers in the range 2.3. We stopped each algorithm when the corresponding quality criterion was below $1\text{E}{-6}$; for Algorithm 4.1 this was the duality gap $m\mu$, whereas SDPT3 and QSDP divide the duality gap by a non-constant factor greater than 1. Thus the quality criterion for our proposed method was more difficult with the caveat that duality gaps for the SOCP reformulation are not directly comparable to the original problem.

We also tested a first-order method of the augmented Lagrangian or alternating directions (ADMM) type [10], these being popular in inverse problems research, which also motivated our efforts on the proposed method. We specifically picked the primal-dual method of Chambolle and Pock [4] due to our experience with the algorithm [41, 40]. It is classified in [6] as the modified primal-dual hybrid gradient method (PDHGM), and can be seen as a preconditioned ADMM. To be able to employ the algorithm, we reformulated (P) into the saddle-point form

$$\min_{\mathbf{X}\in\mathcal{S}_m}\max_{\mathbf{y}\in\mathbb{R}^N} G(\mathbf{X})+\langle\mathbf{y},\mathbf{B}\mathbf{X}\rangle-F^*(\mathbf{y})$$

for $G(\mathbf{X})=\delta_{\mathcal{S}_m^+}(\mathbf{X})$ the ($\{0,\infty\}$-valued) indicator function of $\mathcal{S}_m^+$, and $F^*(\mathbf{y})=\langle\mathbf{y},\mathbf{c}\rangle+\|\mathbf{y}\|_2^2/2$. With these reformulation each step of the PDHGM consists of simple summations, applications of $\mathbf{B}$ and $\mathbf{B}^*$, and projections into $\mathcal{S}_m^+$. As such, the resulting method is comparable to the ADMM approaches in [48, 34]. Since F^* is strongly convex, we used the accelerated version [4] of the algorithm, automatically adjusting the primal and dual step lengths τ and σ, and the inertia parameter θ. This version of the algorithm has $O(1/i^2)$ convergence rate. As the initial step lengths we chose $\tau=1.9/\|\mathbf{B}\|$ and $\sigma=0.5/\|\mathbf{B}\|$. For the stopping criterion, we used $\|\mathbf{X}^i-\mathbf{X}^{i+1}\|/\|\mathbf{X}^1-\mathbf{X}^0\|\leq 1\text{E}{-6}$. We would have preferred a pseudo-duality gap as in [41], but could not observe its consistent convergence below a reasonable gap. We also restricted the maximum number of iterations to 1000000.

Our results are as follows. The time spent in seconds by each of the methods solving problems of given dimension are displayed in Figure 2a and Table 1. The respective iteration counts are displayed in Figure 2b and Table 2. Unfortunately, we could not reasonably obtain iteration counts from CVX, so the methods accessed through it are excluded, most notably Mosek. The times are averages over 10 samples. For iterations, both median and maximum are reported in the table. Since the stopping criteria are not fully comparable, we also report quality values in Table 3. The values are the maximum and median over the samples of the quantity

$$1\text{E}^4\times\frac{f(\mathbf{X}_{\text{alg}})-f(\mathbf{X}_{\text{best}})}{f(\mathbf{0})}$$

for

$$f(u):=\|\mathbf{B}\mathbf{X}-\mathbf{C}\|_2.$$

Here $\mathbf{X}_{\text{alg}}$ is the solution given by the given algorithm, and $\mathbf{X}_{\text{best}}$ is the best solution obtained by all the algorithms.

As we see, overall the quality of solutions by the above metric is comparable for each of the second-order algorithms. The PDHGM does far more poorly, and judging from the median and maximum solution qualities, very unevenly over the different samples. For most values of m, at the worst, the difference between the best solution and the PDHGM solution is about 2% of $f(\mathbf{0})$. For the other methods, aside from one anomalous case with QSDP, the value is at most 0.0002%, and generally a fraction of this. As a first-order method employing projections, it is often unable to converge fast to a high-quality solution, likely doing short steps near the boundary.

The true quadratic methods (QSDP and our proposed method) have somewhat worse quality for small problems than the SOCP reformulations, but become better for larger problems. QSDP in particular has problems with the quality with some problem instances. With regard to computational time, our proposed Algorithm 4.1 performs clearly the best up to problems with dimension about $m = 10$. This threshold is however dependent on the condition number, and will in principle get lower as the condition number increases; see however the experiments for $m = 3$ in the next paragraph. Of the other methods Mosek performs best for small problem instances. Considering the overhead of CVX-SDPT3 to using SDPT3 directly, we expect that Mosek could offer performance comparable to our method if we accessed the solver more directly. For large problem instances, $m = 20$ and above, QSDP has clearly the best performance. Looking at the CPU time in Figure 2a, our proposed method starts to need seemingly exponentially more time as m increases. Comparing this against the iteration count in Figure 2b however confirms the $O(\sqrt{m})$ complexity with bounded κ. Most of the growth in computational time can therefore be attributed to the simple LU solver for (4.20) that we are using in our implementation. This has higher theoretical complexity of $O(m^6)$. (The operators $\mathbf{M}$, $L(\mathbf{X})$ and $L(\mathbf{S})$ in the system matrix have order $O(m^2)$, which gives the $O(m^6)$ complexity.) Our method also requires much more iterations for small problem instances than the other interior point methods, but these are much cheaper, resulting in better performance. Also the quality of the solutions given by the method is better except for the smallest m, suggesting that the iteration count could be reduced by altering the stopping threshold. Nevertheless, our method seems to fulfill the goal of an efficient method for small problem instances, and could also provide improved performance for large instances through improved linear system solvers and preconditioning techniques.

Dependence on the condition number Before celebrating, we still wish to see how much the condition number $\kappa = \Theta/\theta$ in the theoretical complexity bound $O(\sqrt{m}\kappa^2 \log \rho^{-1})$ affects the performance of the algorithm in practise. We therefore ran several tests with different condition numbers in dimension $m = 3$. The setup for these experiments is the same in the previous paragraph except for keeping m fixed and varying the condition number. We exclude CVX-SDPT3 as the solutions are the same as using SDPT3 directly, and we do not need the CVX overhead comparison anymore. We also exclude PDHGM, as the quality of the solutions is not comparable to the second-order methods. We report the time spent, iterations, and the quality of the results in Figure 3, Table 4, Table 5, and Table 6. We may observe that the performance of our algorithm is consistently the best, the iteration count increases quite slowly, and the computational time is almost unaffected by the condition number until our maximum of $\kappa = 10000$.

Conclusion Based on our experiments, we may conclude that the method indeed performs very well for the intended purpose of "weighted projections to the positive definite cone" for small m. While our algorithm requires somewhat more iterations for convergence than existing interior point methods, the iterations are cheaper, not requiring scalings. This gives better computational times.

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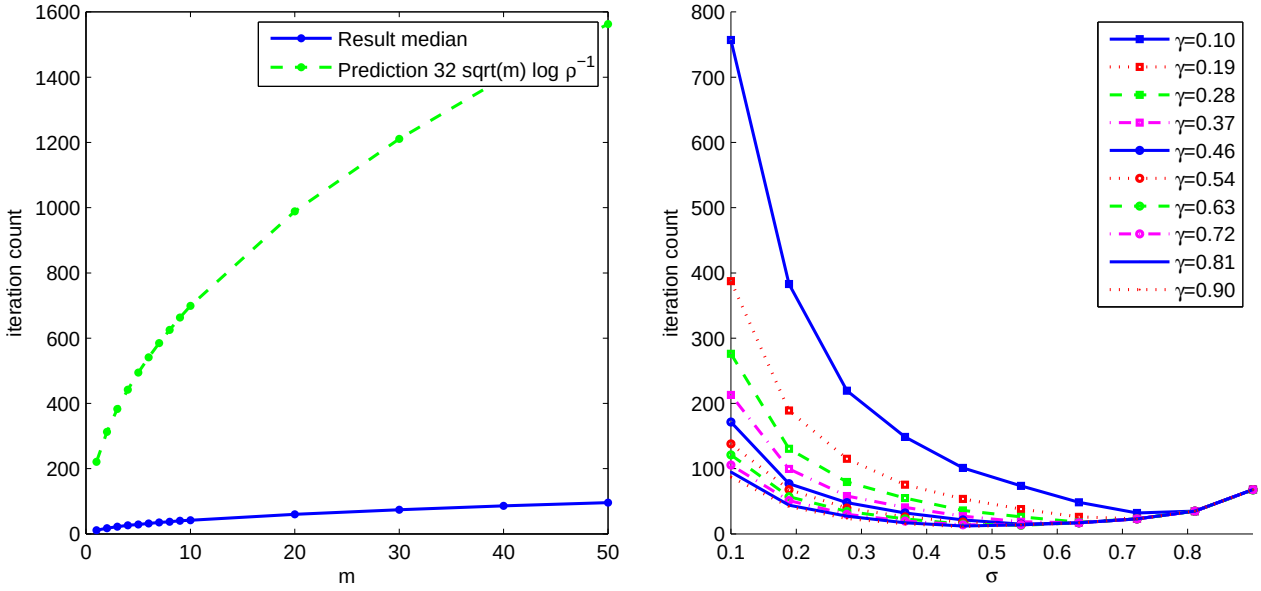
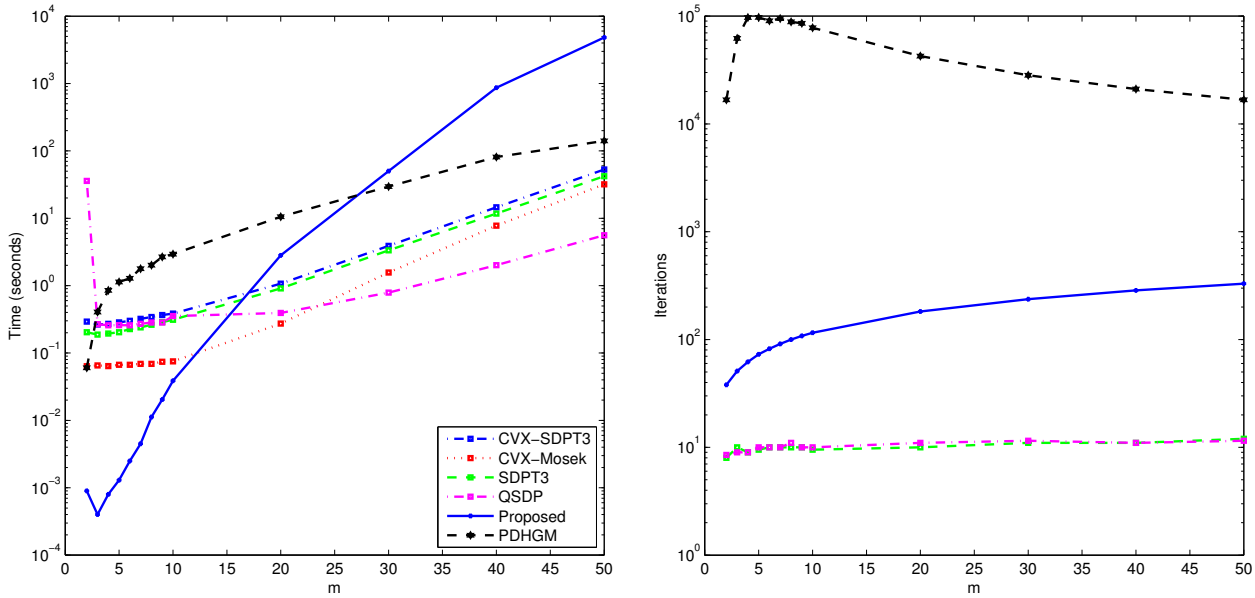


(a) Real and predicted iteration counts for varying m with $\mathbf{M} = \mathbf{I}$ with fixed.

(b) Iteration counts for varying γ and σ and fixed $m = 3$ and non-identity $\mathbf{M}$.

Figure 1: Iteration counts of numerical experiments for $\rho = 0.001$ fractional decrease of μ. Each data point is the median over 10 samples of $\mathbf{C}$.



(a) CPU time in seconds (logarithmic)

(b) Iteration counts (logarithmic)

Figure 2: Comparison of time spent and iterations by different methods on problems of varying dimension m. The quantities reported are averages over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

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Table 1: Comparison of time spent (in seconds) by different methods on problems of varying dimension m. The time is an average over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

m	CVX-SDPT3	CVX-Mosek	SDPT3	QSDP	Proposed	PDHGM
2	0.2924	0.0638	0.2031	35.8996	0.0009	0.0604
3	0.2648	0.0654	0.1876	0.2675	0.0004	0.4066
4	0.2708	0.0639	0.1946	0.2584	0.0008	0.8510
5	0.2850	0.0670	0.2038	0.2591	0.0013	1.1314
6	0.3003	0.0669	0.2267	0.2578	0.0025	1.2878
7	0.3220	0.0691	0.2407	0.2711	0.0045	1.7840
8	0.3428	0.0689	0.2629	0.2840	0.0112	2.0160
9	0.3667	0.0743	0.2869	0.2840	0.0204	2.6710
10	0.3855	0.0753	0.3111	0.3537	0.0387	2.9390
20	1.0694	0.2740	0.9102	0.3938	2.8204	10.5210
30	3.8842	1.5656	3.3356	0.7874	50.1446	29.5960
40	14.5810	7.7680	11.7320	2.0170	865.0580	81.0480
50	52.8980	31.8890	42.1290	5.5870	4816.4030	140.6230

Table 2: Comparison of iterations used by different methods on problems of varying dimension m. We report median and maximum over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

	SDPT3		QSDP		Proposed		PDHGM	
m	med	max	med	max	med	max	med	max
2	8	10	8	1000	38	40	16759	52384
3	10	10	9	11	51	65	62222	87438
4	9	11	9	12	62	64	96972	130456
5	10	10	10	11	73	74	97308	122719
6	10	10	10	11	82	83	90654	105976
7	10	10	10	12	91	92	94828	105544
8	10	10	11	12	100	101	88570	95893
9	10	11	10	12	108	109	85806	87630
10	10	10	10	11	116	116	77956	82716
20	10	12	11	12	182	183	42561	42809
30	11	12	12	13	237	238	28300	28420
40	11	12	11	12	286	286	21086	21126
50	12	13	12	12	330	330	16746	16763

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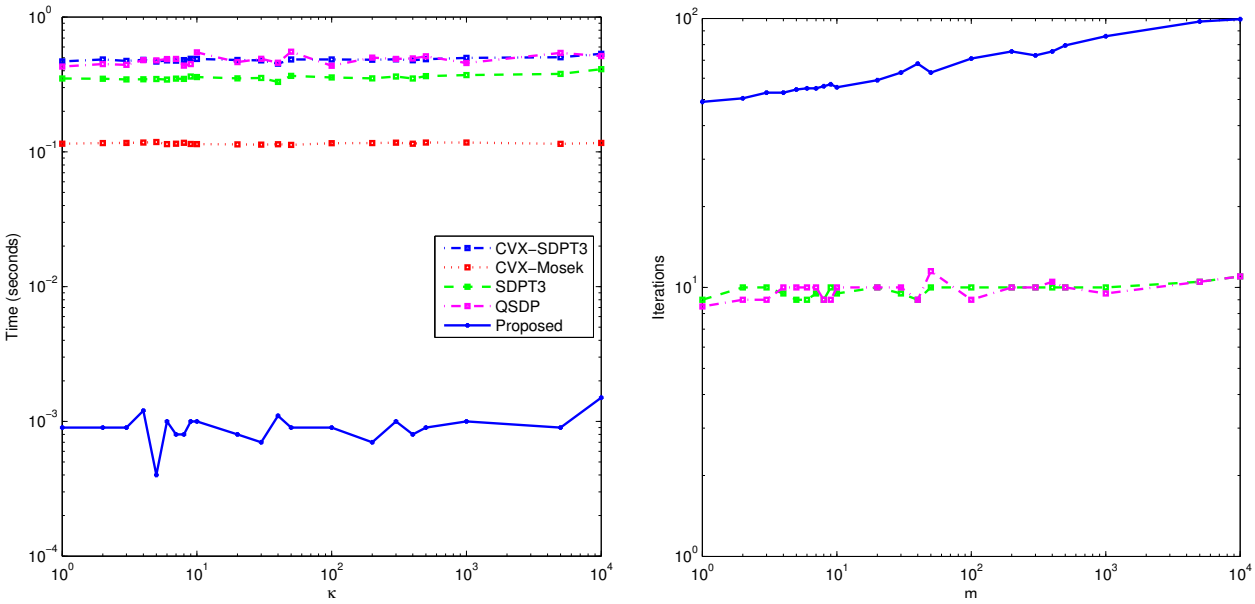
Table 3: Comparison of quality of solution by different methods on problems of varying dimension m. The values reported are the maximum and median over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$ of the quantity $1\text{E}4 \times (f(\mathbf{X}_{\text{alg}}) - f(\mathbf{X}_{\text{best}}))/f(\mathbf{0})$, for $f(u) = \|\mathbf{B}\mathbf{X} - \mathbf{C}\|_2$.

	CVX-SDPT3		CVX-Mosek		SDPT3		QSDP		Proposed		PDHGM	
m	med	max	med	max	med	max	med	max	med	max	med	max
2	0	0.019	0.004	0.017	0	0.019	0.003	1705.747	0.006	0.021	1.091	479.973
3	0	0.008	0.006	0.026	0	0.008	0.001	2.034	0.003	1.308	36.312	2139.807
4	0.004	0.009	0.005	0.038	0.004	0.009	0	0.003	0	0.002	94.306	472.549
5	0.002	0.009	0.007	0.042	0.002	0.009	0	0.002	0	0.002	302.498	1170.420
6	0.002	0.013	0.007	0.081	0.002	0.013	0	0	0	0	599.759	1189.876
7	0.001	0.004	0.015	0.073	0.001	0.004	0	0	0	0	602.551	1362.774
8	0.001	0.005	0.044	0.088	0.001	0.005	0	0	0	0	928.543	1321.088
9	0	0.005	0.021	0.071	0	0.005	0	0.002	0	0	975.373	1812.893
10	0.001	0.006	0.028	0.156	0.001	0.006	0	0.001	0	0	1233.966	1897.175
20	0.002	0.006	0.043	0.172	0.002	0.006	0	0.002	0	0	1711.373	1966.149
30	0	0.003	0.053	0.216	0	0.003	0	0.003	0	0	1798.433	2521.595
40	0.004	0.009	0.099	0.246	0.004	0.009	0	0.002	0	0	1909.991	2102.644
50	0.002	0.007	0.155	0.257	0.002	0.007	0	0.001	0	0	1867.902	2118.766



(a) CPU time in seconds (both axes logarithmic) **(b)** Iteration counts (both axes logarithmic)

Figure 3: Comparison of time spent (in seconds) and iterations by different methods on problems with varying condition number κ of $\mathbf{M}$ with dimension $m = 3$. The time reported is an average over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

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Table 4: Comparison of time spent (in seconds) by different methods on problems with varying condition number κ of $\mathbf{M}$ with dimension $m = 3$. We report the average over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

κ	CVX-Mosek	SDPT3	QSDP	Proposed
1	0.1150	0.3504	0.4303	0.0009
2	0.1162	0.3491	0.4488	0.0009
3	0.1165	0.3452	0.4425	0.0009
4	0.1174	0.3458	0.4799	0.0012
5	0.1184	0.3472	0.4773	0.0004
10	0.1142	0.3590	0.5474	0.0010
50	0.1127	0.3663	0.5534	0.0009
100	0.1161	0.3569	0.4353	0.0009
500	0.1173	0.3647	0.5121	0.0009
1000	0.1173	0.3718	0.4564	0.0010
5000	0.1148	0.3788	0.5425	0.0009
10000	0.1165	0.4108	0.5143	0.0015

Table 5: Comparison of iterations used by different methods on problems with varying condition number κ of $\mathbf{M}$ with dimension $m = 3$. We report median and maximum over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$.

	SDPT3		QSDP		Proposed	
κ	med	max	med	max	med	max
1	9	10	8	11	49	51
2	10	11	9	11	50	52
3	10	10	9	10	53	58
4	10	10	10	11	53	56
5	9	10	10	12	54	57
10	10	12	10	12	56	61
50	10	12	12	14	63	70
100	10	11	9	11	71	74
500	10	11	10	12	80	84
1000	10	11	10	12	86	93
5000	10	12	10	13	98	103
10000	11	13	11	14	100	105

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Table 6: Comparison of quality of solution by different methods on problems with varying condition number κ of $\mathbf{M}$ with dimension $m = 3$. The values reported are the maximum and median over 10 samples of random operator $\mathbf{B} : \mathcal{S}_m \to \mathcal{S}_m$ and random $\mathbf{C} \in \mathcal{S}_m$ of the quantity $1\text{E}^4 \times (f(\mathbf{X}_{\text{alg}}) - f(\mathbf{X}_{\text{best}}))/f(\mathbf{0})$, for $f(u) = \|\mathbf{BX} - \mathbf{C}\|_2$.

	CVX-Mosek		SDPT3		QSDP		Proposed	
κ	med	max	med	max	med	max	med	max
1	0.002	0.032	0	0.010	0.001	0.018	0.002	0.015
2	0.003	0.018	0	0.005	0.001	0.011	0.002	0.007
3	0.003	0.031	0	0.003	0.001	0.018	0.004	0.045
4	0.004	0.014	0	0.010	0	0.006	0.002	0.005
5	0.002	0.015	0.002	0.008	0	26790.295	0.002	0.023
10	0.003	0.023	0.002	0.010	0	0.003	0.003	0.012
50	0	0.019	0	0.009	0	0.004	0.003	0.022
100	0.001	0.006	0	0.014	0.001	0.007	0.003	0.009
500	0.001	0.010	0.003	0.009	0	0.007	0.003	0.006
1000	0	0.011	0.003	0.007	0	657662.704	0.003	0.014
5000	0	0.016	0.002	0.011	0.003	115604.491	0.003	0.006
10000	0	0.007	0.001	0.010	0	0.007	0.003	0.008

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