INVERSE PROBLEMS WITH SECOND-ORDER TOTAL GENERALIZED VARIATION CONSTRAINTS

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ABSTRACT

Total Generalized Variation (TGV) has recently been introduced as penalty functional for modelling images with edges as well as smooth variations [2]. It can be interpreted as a "sparse" penalization of optimal balancing from the first up to the k-th distributional derivative and leads to desirable results when applied to image denoising, i.e., L^2-fitting with TGV penalty. The present paper studies TGV of second order in the context of solving ill-posed linear inverse problems. Existence and stability for solutions of Tikhonov-functional minimization with respect to the data is shown and applied to the problem of recovering an image from blurred and noisy data.

Keywords— Total Generalized Variation, linear inverse problems, Tikhonov regularization, deblurring problem.

1. INTRODUCTION

Most mathematical formulations of inverse problems, in particular of mathematical imaging problems are cast in the form of minimizing a Tikhonov functional, i.e.,

$$\min_u \; F(u) + \alpha R(u)$$

where F represents the fidelity with respect to the measured data, R is a regularization functional and $\alpha > 0$ a parameter. With a linear, continuous and usually ill-posed forward operator K as well as possibly error-prone data f, the data fidelity term is commonly chosen as

$$F(u) = \frac{\|Ku - f\|^2}{2}$$

with a Hilbert-space norm $\|\cdot\|$. For imaging problems, popular choices for the regularization functionals are one-homogeneous functionals, in particular, the Total Variation seminorm [9]

$$R(u) = \int_\Omega \,\mathrm{d}|Du| = \|Du\|_{\mathcal{M}}$$

where $|Du|$ denotes the variation-measure of the distributional derivative Du which is a vector-valued Radon measure. It allows for discontinuities to appear along hypersurfaces and therefore yields a suitable model for images with edges. Unfortunately, for true data containing smooth regions, Total Variation regularization tends to produce undesired piecewise constant solutions; a phenomenon which is known as the "staircasing effect" [8, 11]. To overcome this problem, higher-order functionals have been proposed [3, 5], for instance the weighted infimal convolution of the first- and second order total-variation, i.e., $\int_\Omega \,\mathrm{d}|Du|$ and $\int_\Omega \,\mathrm{d}|D^2u|$. With such regularizers, a reduction of the staircasing effect can be observed. However, it may still occur, usually in the neighborhood of edges.

In [2], the *Total Generalized Variation* (TGV) of order k, defined as

$$\mathrm{TGV}_\alpha^k(u) = \sup\Big\{ \int_\Omega u \,\mathrm{div}^k v \,\mathrm{d}x \;\Big|\; v \in \mathcal{C}_c^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)), \\ \|\mathrm{div}^l v\|_\infty \le \alpha_l,\; l = 0, \ldots, k-1 \Big\}, \quad (1)$$

has been proposed and analyzed. It constitutes a new image model which can be interpreted to incorporate smoothness from the first up to the k-th derivative. Here, $\mathrm{Sym}^k(\mathbb{R}^d)$ denotes the space of symmetric tensors of order k with arguments in $\mathbb{R}^d$ and $\alpha_l > 0$ are fixed parameters. Choosing $k = 1$ and $\alpha_0 = 1$ yields the usual Total Variation functional. It is immediate that in $L^2(\Omega)$, the denoising problem which corresponds to $K = I$, TGV_α^2 as a regularization term leads to a well-posed minimization problem for the Tikhonov functional. The numerical experiments carried out in [2] show that TGV_α^2 produces visually appealing results with almost no staircase effect present in the solution (see Figure 1).

When solving ill-posed inverse problems for $K : L^2(\Omega) \to L^2(\Omega)$ with a Total Generalized Variation regularization, however, it is not immediate that the problem of minimizing the Tikhonov functional is well-posed. The present paper addresses this issue by analyzing the case of $k = 2$, i.e., Total Generalized Variation of second order. We will show that TGV_α^2 is a seminorm on $\mathrm{BV}(\Omega)$ and that $\|u\|_1 + \mathrm{TGV}_\alpha^2(u)$ is topologically equivalent to the BV-norm. Based on this result, existence and well-posedness of TGV_α^2-regularization for Tikhonov functionals is derived. Moreover, we apply these results to solve an image deblurring problem. Finally, numerical experiments illustrate the feasibility of Total Generalized Variation regularization of second order.

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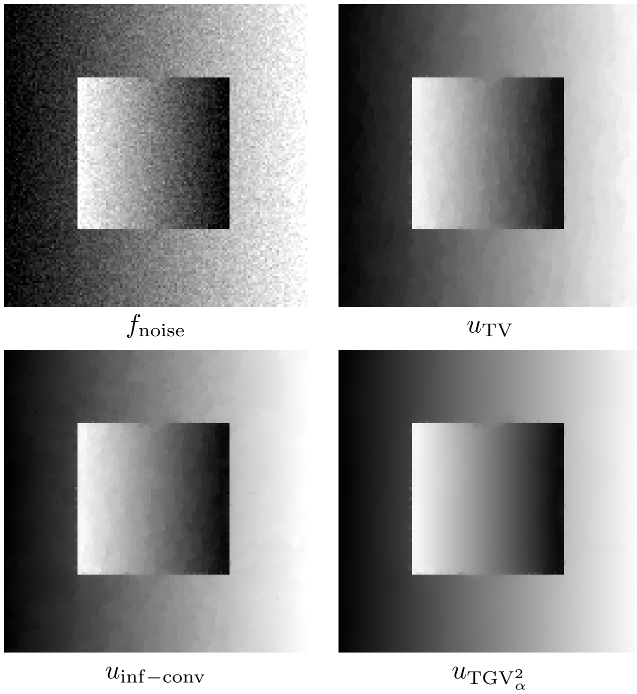

Fig. 1. Denoising with Total Variation, infimal convolution and Total Generalized Variation.

2. BASIC PROPERTIES OF SECOND-ORDER TGV

Let us first define Total Generalized Variation of second order as well as mention some basic properties.

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\alpha = (\alpha_0, \alpha_1) > 0$. The functional assigning each $u \in L^1_{\text{loc}}(\Omega)$ the value

$$\mathrm{TGV}^2_\alpha(u) = \sup\Big\{ \int_\Omega u \operatorname{div}^2 v \,\mathrm{d}x \;\Big|\; v \in \mathcal{C}^2_{\mathrm{c}}(\Omega, S^{d\times d}), \\ \|v\|_\infty \leq \alpha_0,\ \|\operatorname{div} v\|_\infty \leq \alpha_1 \Big\} \quad (2)$$

is called the *Total Generalized Variation* of second order.

Here, $S^{d\times d}$ is the set of symmetric matrices, $\mathcal{C}^2_{\mathrm{c}}(\Omega, S^{d\times d})$ the vector space of compactly supported, twice continuously differentiable $S^{d\times d}$-valued mappings and $\operatorname{div} v \in \mathcal{C}^1_{\mathrm{c}}(\Omega, \mathbb{R}^d)$, $\operatorname{div}^2 v \in \mathcal{C}_{\mathrm{c}}(\Omega)$ is defined by

$$(\operatorname{div} v)_i = \sum_{j=1}^d \frac{\partial v_{ij}}{\partial x_j}, \quad \operatorname{div}^2 v = \sum_{i=1}^d \frac{\partial^2 v_{ii}}{\partial x_i^2} + 2\sum_{i<j} \frac{\partial^2 v_{ij}}{\partial x_i \partial x_j}.$$

The norms of $v \in \mathcal{C}_{\mathrm{c}}(\Omega, S^{d\times d})$, $\omega \in \mathcal{C}_{\mathrm{c}}(\Omega, \mathbb{R}^d)$ are given by

$$\|v\|_\infty = \sup_{x\in\Omega} \Big(\sum_{i=1}^d |v_{ii}(x)|^2 + 2\sum_{i<j} |v_{ij}(x)|^2 \Big)^{1/2},$$

$$\|\omega\|_\infty = \sup_{x\in\Omega} \Big(\sum_{i=1}^d |\omega_i(x)|^2 \Big)^{1/2}.$$

The space

$$\mathrm{BGV}^2_\alpha(\Omega) = \{ u \in L^1(\Omega) \;\big|\; \mathrm{TGV}^2_\alpha(u) < \infty \}$$

equipped with the norm

$$\|u\|_{\mathrm{BGV}^2_\alpha} = \|u\|_1 + \mathrm{TGV}^2_\alpha(u)$$

is called the space of functions of *Bounded Generalized Variation* of order 2.

Basic results about this functional obtained in [2] can be summarized as follows.

Theorem 2.2. *Total Generalized Variation of second order enjoys the following properties:*

1. TGV^2_α *is a semi-norm on the Banach space* $\mathrm{BGV}^2_\alpha(\Omega)$,
2. $\mathrm{TGV}^2_\alpha(u) = 0$ *if and only if* u *is a polynomial of degree less than* 2,
3. TGV^2_α *and* $\mathrm{TGV}^2_{\tilde\alpha}$ *are equivalent for* $\tilde\alpha = (\tilde\alpha_0, \tilde\alpha_1) > 0$,
4. TGV^2_α *is rotationally invariant,*
5. TGV^2_α *satisfies, for* $r > 0$ *and* $(\rho_r u)(x) = u(rx)$, *the scaling property*

$$\mathrm{TGV}^2_\alpha \circ \rho_r = r^{-d}\, \mathrm{TGV}^2_{\tilde\alpha}(u), \ \ (\tilde\alpha_0, \tilde\alpha_1) = (\alpha_0 r^2, \alpha_1 r),$$

6. TGV^2_α *is proper, convex and lower semi-continuous on each* $L^p(\Omega)$, $1 \leq p < \infty$.

3. TOPOLOGICAL EQUIVALENCE WITH BV

To obtain existence results for Tikhonov functionals with TGV^2_α-penalty, we reduce the functional-analytic setting to the space $\mathrm{BV}(\Omega)$ by establishing topological equivalence. This will be done in two steps.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain. For each* $u \in L^1(\Omega)$ *we have*

$$\mathrm{TGV}^2_\alpha(u) = \min_{w \in \mathrm{BD}(\Omega)} \ \alpha_1 \|Du - w\|_{\mathcal{M}} + \alpha_0 \|\mathcal{E}w\|_{\mathcal{M}} \quad (3)$$

where $\mathrm{BD}(\Omega)$ *denotes the space of vector fields of* Bounded Deformation *[10], i.e.,* $w \in L^1(\Omega, \mathbb{R}^d)$ *such that the distributional symmetrized derivative* $\mathcal{E}w = \frac{1}{2}(\nabla w + \nabla w^{\mathrm{T}})$ *is a* $S^{d\times d}$*-valued Radon measure.*

Sketch of proof. Choosing $X = \mathcal{C}^2_0(\Omega, S^{d\times d})$, $Y = \mathcal{C}^1_0(\Omega, \mathbb{R}^d)$, $\Lambda = \operatorname{div} \in \mathcal{L}\big(X, Y\big)$ and, for $v \in X$, $\omega \in Y$,

$$F_1(v) = I_{\{\|\cdot\|_\infty \leq \alpha_0\}}(v),$$

$$F_2(\omega) = I_{\{\|\cdot\|_\infty \leq \alpha_1\}}(\omega) - \int_\Omega u \operatorname{div} \omega \,\mathrm{d}x$$

we see with density arguments that

$$\mathrm{TGV}^2_\alpha(u) = -\inf_{v\in X} \ F_1(v) + F_2(\Lambda v).$$

Furthermore, $Y = \bigcup_{\lambda \geq 0} \lambda\big(\mathrm{dom}(F_1) - \Lambda\, \mathrm{dom}(F_2)\big)$, hence by Fenchel-Rockafellar duality [1], it follows that

$$\mathrm{TGV}^2_\alpha(u) = \min_{w \in Y^*} \ F_1^*(-\Lambda^* w) + F_2^*(w).$$

Now, $Y^* = \mathcal{C}_0^1(\Omega, \mathbb{R}^d)^*$ can be regarded as a space of distributions and the dual functionals can be written as

$$F_1^*(-\Lambda^* w) = \begin{cases} \alpha_0 \|\mathcal{E}w\|_{\mathcal{M}} & \text{if } w \in \mathrm{BD}(\Omega) \\ \infty & \text{else,} \end{cases}$$

$$F_2^*(w) = \begin{cases} \alpha_1 \|Du - w\|_{\mathcal{M}} & \text{if } Du - w \in \mathcal{M}(\Omega, \mathbb{R}^d) \\ \infty & \text{else.} \end{cases}$$

Since $\mathrm{BD}(\Omega) \subset \mathcal{M}(\Omega, \mathbb{R}^d)$, the result follows. □

Remark 3.2. The minimization in (3) can be interpreted as an optimal balancing between the first and second derivative of u in terms of "sparse" penalization (via the Radon norm).

The second step combines (3) with the Sobolev-Korn inequality for vector fields of Bounded Deformation.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exist constants $0 < c < C < \infty$ such that for each $u \in \mathrm{BGV}_\alpha^2(\Omega)$ there holds*

$$c\|u\|_{\mathrm{BV}} \le \|u\|_1 + \mathrm{TGV}_\alpha^2(u) \le C\|u\|_{\mathrm{BV}}.$$

Proof. Setting $w = 0$ in (3) immediately implies that for each $u \in \mathrm{BGV}_\alpha^2(\Omega)$ we have $\mathrm{TGV}_\alpha^2(u) \le \alpha_1 \mathrm{TV}(u)$, hence we can set $C = \max(1, \alpha_1)$.

On the other hand, we may assume that $Du \in \mathcal{M}(\Omega, \mathbb{R}^d)$ since otherwise, $\|Du - w\|_{\mathcal{M}} = \infty$ for all $w \in \mathrm{BD}(\Omega)$ and hence, $\mathrm{TGV}_\alpha^2(u) = \infty$ by (3). Observe that $\bar{w} \in \ker \mathcal{E}$ if and only if $\bar{w}(x) = Ax + b$ for some $A \in \mathbb{R}^{d\times d}$ satisfying $A^{\mathrm{T}} = -A$ and $b \in \mathbb{R}^d$. We show that there is a $C_1 > 0$ such that for each $u \in \mathrm{BV}(\Omega)$ and $\bar{w} \in \ker \mathcal{E}$ there holds

$$\|Du\|_{\mathcal{M}} \le C_1\big(\|Du - \bar{w}\|_{\mathcal{M}} + \|u\|_1\big). \tag{4}$$

Suppose that this is not the case. Then, there exist sequences $\{u^n\}$ in $\mathrm{BV}(\Omega)$ and $\{\bar{w}^n\}$ in $\ker \mathcal{E}$ such that

$$\|Du^n\|_{\mathcal{M}} = 1, \quad \|Du^n - \bar{w}^n\|_{\mathcal{M}} + \|u^n\|_1 \le \tfrac{1}{n}$$

for all $n \in \mathbb{N}$. Consequently, $\lim_{n\to\infty} u^n = 0$ in $L^1(\Omega)$, $\lim_{n\to\infty} Du^n - \bar{w}^n = 0$ in $\mathcal{M}(\Omega, \mathbb{R}^d)$ and $\{\bar{w}^n\}$ is bounded in $\ker \mathcal{E}$. Since the latter is finite-dimensional, there exists a convergent subsequence, i.e., $\lim_{k\to\infty} \bar{w}^{n_k} = w$ for some $w \in \ker \mathcal{E}$. It follows that $\lim_{k\to\infty} Du^{n_k} = w$, thus $w = 0$ by closedness of the distributional derivative. This means in particular $\lim_{k\to\infty} \|Du^{n_k}\|_{\mathcal{M}} = 0$ which is a contradiction since each $\|Du^{n_k}\|_{\mathcal{M}} = 1$.

Next, recall that a Sobolev-Korn inequality holds for $\mathrm{BD}(\Omega)$ [10]: There is a $C_2 > 0$ such that for each $w \in \mathrm{BD}(\Omega)$ there exists a $\bar{w} \in \ker \mathcal{E}$ such that $\|w - \bar{w}\|_1 \le C_2\|\mathcal{E}w\|_{\mathcal{M}}$. For this $\bar{w}$, we have, for some $C_3 > 0$,

$$\begin{aligned}\|Du - \bar{w}\|_{\mathcal{M}} &\le \|Du - w\|_{\mathcal{M}} + \|w - \bar{w}\|_1 \\ &\le C_3\big(\alpha_1\|Du - w\|_{\mathcal{M}} + \alpha_0\|\mathcal{E}w\|_{\mathcal{M}}\big).\end{aligned}$$

Plugged into (4) and adding $\|u\|_1$ on both sides, it follows that the inequality

$$\|u\|_{\mathrm{BV}} \le C_4\big(\|u\|_1 + \alpha_1\|Du - w\|_{\mathcal{M}} + \alpha_0\|\mathcal{E}w\|_{\mathcal{M}}\big)$$

holds for some $C_4 > 0$ independent of u and w. Taking the minimum over all $w \in \mathrm{BD}(\Omega)$ and choosing $c = C_4^{-1}$ finally yields the result by virtue of (3). □

Since both $\mathrm{BV}(\Omega)$ and $\mathrm{BGV}_\alpha^2(\Omega)$ are Banach spaces, we immediately have:

Corollary 3.4. *If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz-domain, then $\mathrm{BGV}_\alpha^2(\Omega) = \mathrm{BV}(\Omega)$ for all $(\alpha_0, \alpha_1) > 0$ in the sense of topologically equivalent Banach spaces.*

4. EXISTENCE AND STABILITY OF SOLUTIONS

Let, in the following, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. The coercivity needed for showing existence of solutions for the Tikhonov functional is implied by the following inequality of Poincaré-Wirtinger type.

Proposition 4.1. *Let $1 < p < \infty$ such that $p \le d/(d-1)$ and $P : L^p(\Omega) \to \mathcal{P}^1(\Omega)$ a linear projection onto the space of affine functions $\mathcal{P}^1(\Omega)$. Then, there is a $C > 0$ such that*

$$\|u\|_p \le C\,\mathrm{TGV}_\alpha^2(u) \qquad \forall u \in \ker P \subset L^p(\Omega). \tag{5}$$

Proof. If this is not true, there exists a sequence $\{u^n\}$ in $\ker P$ with $\|u^n\|_p = 1$ such that $1 \ge C(n)\,\mathrm{TGV}_\alpha^2(u^n)$ where $C(n) \ge n$. We may assume $u^n \rightharpoonup u$ in $L^p(\Omega)$ with $u \in \ker P$. According to Theorem 3.3, $\{u^n\}$ is also bounded in $\mathrm{BV}(\Omega)$, thus we also have, by compact embedding, that $\lim_{n\to\infty} u^n = u$ in $L^1(\Omega)$. Lower semi-continuity now implies $\mathrm{TGV}_\alpha^2(u) \le \liminf_{n\to\infty} \mathrm{TGV}_\alpha^2(u^n) = 0$, hence $u \in \mathcal{P}^1(\Omega) \cap \ker P$ (see Theorem 2.2) and, consequently, $u = 0$. Thus, $u^n \to 0$ in $\mathrm{BV}(\Omega)$ and by continuous embedding, also in $L^p(\Omega)$, which is a contradiction to $\|u^n\|_p = 1$ for all n. □

Theorem 4.2. *Let $1 < p < \infty$, $p \le d/(d-1)$, Y be a Hilbert space, $K \in \mathcal{L}\big(L^p(\Omega), Y\big)$ a linear and continuous operator which is injective on $\mathcal{P}^1(\Omega)$ and $f \in Y$. Then, the problem*

$$\min_{u \in L^p(\Omega)} \ \tfrac{1}{2}\|Ku - f\|^2 + \mathrm{TGV}_\alpha^2(u) \tag{6}$$

admits a solution.

Proof. Choose P according to Proposition 4.1 (such a P exists since $\mathcal{P}^1(\Omega)$ is finite-dimensional) as well as a minimizing sequence $\{u^n\}$. By $\mathrm{TGV}_\alpha^2(u) = \mathrm{TGV}_\alpha^2(u - Pu)$ (since $\mathrm{TGV}_\alpha^2(Pu) = 0$, see Theorem 2.2) and (5), $\{u^n - Pu^n\}$ is bounded in $L^p(\Omega)$. Moreover, $\{\|Ku^n - f\|\}$ is bounded. Since K is injective on the finite-dimensional space $\mathcal{P}^1(\Omega)$, there is a $C_1 > 0$ such that $\|Pu\|_p \le C_1\|KPu\|$, hence

$$\begin{aligned}\|Pu^n\|_p &\le C_1\|KPu^n\| \\ &\le C_1\big(\|Ku^n - f\| + \|K(u^n - Pu^n) - f\|\big) \le C_2,\end{aligned}$$

for some $C_2 > 0$ implying that $\{u^n\}$ is bounded in $L^p(\Omega)$. Thus, there exists a weakly convergent subsequence with limit u^* which can be seen to be a minimizer by weak lower semi-continuity (also confer Theorem 2.2). □

Theorem 4.3. *In the situation of Theorem 4.2, let $\{f^n\}$ be a sequence in Y with $\lim_{n\to\infty} f^n = f$. Then each sequence $\{u^n\}$ of minimizers u^n of* (6) *with data f^n is relatively weakly compact in $L^p(\Omega)$ and each weak accumulation point $u^* = \lim_{k\to\infty} u^{n_k}$ minimizes* (6) *with data f with $\lim_{k\to\infty} = \mathrm{TGV}^2_\alpha(u^{n_k}) = \mathrm{TGV}^2_\alpha(u^*)$.*

Proof. The proof follows the lines of [6]. Denote by F_n and F the functional to minimize in (6) with data f^n and f, respectively. Then, for any $u \in \mathrm{BV}(\Omega)$, $F_n(u^n) \leq F_n(u)$, hence, using that $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$ for $a, b \in Y$,

$$\begin{aligned} F(u^n) &\leq \|Ku^n - f^n\|^2 + \mathrm{TGV}^2_\alpha(u^n) + \|f^n - f\|^2 \\ &\leq 2F_n(u^n) + \|f^n - f\|^2 \leq 2F_n(u) + \|f^n - f\|^2. \end{aligned}$$

This shows that $\{F(u^n)\}$ is bounded and by the same arguments as in Theorem 4.2, it follows that $\{u^n\}$ is bounded in $L^p(\Omega)$ and therefore relatively weakly compact. Now, let u^* be a weak accumulation point, i.e., $u^{n_k} \rightharpoonup u^*$. It follows by weak lower semi-continuity that $\frac{1}{2}\|Ku^* - f\|^2 \leq \liminf_{k\to\infty} \frac{1}{2}\|Ku^{n_k} - f^{n_k}\|^2$ and $\mathrm{TGV}^2_\alpha(u^*) \leq \liminf_{k\to\infty} \mathrm{TGV}^2_\alpha(u^{n_k})$. Hence,

$$\begin{aligned} F(u^*) &\leq \liminf_{k\to\infty} F_{n_k}(u^{n_k}) \leq \limsup_{k\to\infty} F_{n_k}(u^{n_k}) \\ &\leq \lim_{k\to\infty} F_{n_k}(u) = F(u) \end{aligned}$$

for each $u \in \mathrm{BV}(\Omega)$, showing that u^* is a minimizer. In particular, plugging in u^* leads to $\lim_{k\to\infty} F_{n_k}(u^{n_k}) = F(u^*)$. Now, $\limsup_{k\to\infty} \mathrm{TGV}^2_\alpha(u^{n_k}) > \mathrm{TGV}^2_\alpha(u^*)$ would contradict this, hence $\mathrm{TGV}^2_\alpha(u^{n_k}) \to \mathrm{TGV}^2_\alpha(u^*)$. □

5. APPLICATION TO DECONVOLUTION

Let $d = 2$ and $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Pick a blurring kernel $k \in L^1(\Omega_0)$ satisfying $\bar{k} = \int_{\Omega_0} k \,\mathrm{d}x \neq 0$. Moreover, let $\Omega' \subset \mathbb{R}^2$ be a domain with $\Omega' - \Omega_0 \subset \Omega$. Then

$$(Ku)(x) = \int_{\Omega_0} u(x-y)k(y)\,\mathrm{d}y, \quad x \in \Omega'$$

fulfills $K \in \mathcal{L}\big(L^2(\Omega), L^2(\Omega')\big)$. If m denotes the vector with components $m_i = \int_{\Omega_0} y_i k(y)\,\mathrm{d}y$, then an affine function $u(x) = a \cdot x + b$ with $Ku = 0$ satisfies

$$\int_{\Omega_0} \big(a\cdot(x-y) + b\big)k(y)\,\mathrm{d}y = a\bar{k}\cdot x + b\bar{k} - a\cdot m = 0$$

for all $x \in \Omega'$. Since this is a domain, $a\bar{k} = 0$ and $b\bar{k} - a\cdot m = 0$ which implies $a = 0$ and $b = 0$. Hence K is injective on $\mathcal{P}^1(\Omega)$.

If $f \in L^2(\Omega')$ is a noise-contaminated image blurred by the convolution operator K, then according to Theorem 4.2, the TGV^2_α-based Tikhonov regularization

$$\min_{u\in L^2(\Omega)} \frac{1}{2}\int_{\Omega'} |(u*k)(x) - f(x)|^2\,\mathrm{d}x + \mathrm{TGV}^2_\alpha(u)$$

admits a solution which stably depends on f (Theorem 4.3).

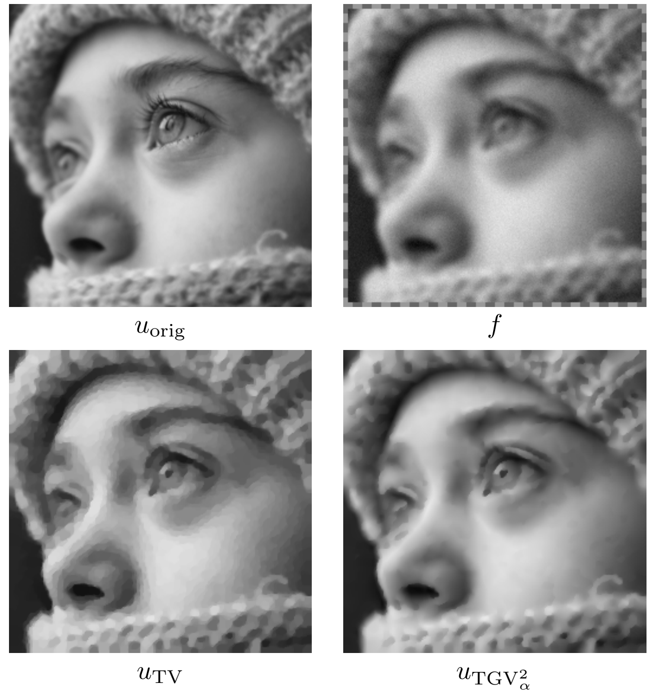

Fig. 2. Deconvolution example. The original image u_{orig} [7] has been blurred and contaminated by noise resulting in f, u_{TV} and $u_{\mathrm{TGV}^2_\alpha}$ are the regularized solutions recovered from f.

This convex minimization problem can be put, for instance, in a saddle-point formulation and solved numerically by a primal-dual algorithm [4]. Figure 2 shows the effect of TGV^2_α compared to TV for deblurring a sample image.

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