Transport equation and image interpolation with SBD velocity fields

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Abstract

We consider an extended formulation of the transport equation that remains meaningful with discontinuous velocity fields b, assuming that $(1, b)$ is a special function of bounded deformation (SBD). We study existence, uniqueness, and continuity/stability of the presented formulation. We then apply this study to the problem of fitting to available data a space-time image subject to the optical flow constraint. Moreover, in order to carry out these studies, we refine the SBD approximation theorem of Chambolle to show the convergence of traces.

Resumé

Nous considérons une extension de l'équation de transport qui reste valide avec des champs de vitesses discontinues b, en supposant que $(1, b)$ est une fonction spécial de déformation bornée (SBD 'special function of bounded deformation' en anglais). Nous étudions l'existence, l'unicité et de la continuité/stabilité du modèle présenté. Nous appliquons ensuite ces résultats dans le problème de l'ajustement d'une image sur l'espace-temps aux données disponibles, sous la contrainte du flux optique. En outre, a fin de conclure ces études, on perfectionne la théorème d'approximation des SBD par Chambolle pour montrer la convergence des traces.

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1. Introduction

Our primary objective in this work is to extend the *transport equation* to model both jump sources and sinks. We assume that $u = (1, b)$ is a special function of bounded deformation (SBD; see Temam [1] and Ambrosio et al. [2]), supported on $\mathrm{cl}((0, T) \times \Omega) \subset \mathbb{R}^{n+1}$. We then ask for the existence of $I : (0, T) \times \Omega \to \mathbb{R}$ and $\tau : J_u \to \mathbb{R}$, defined on the ($\mathcal{H}^n$-rectifiable) jump set of u, satisfying the distributional equation

$$\mathrm{Div}(Iu) - I \operatorname{div} u\mathcal{L}^{n+1} - \tau \operatorname{Div}^j u = 0 \quad \text{on } \mathbb{R}^{n+1}. \tag{1}$$

Constraints may be placed on the one-sided traces of I on parts of J_u, including an initial condition at time $t = 0$. We denote by $\operatorname{div} u$ and $\operatorname{Div}^j u$, respectively, the absolutely continuous and jump

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parts of the distributional divergence $\operatorname{Div} u$. Note that the first two terms of (1) reduce to $\langle \nabla I, u\rangle$ when everything is smooth.

To motivate (1), in particular the introduction of the term $\tau \operatorname{Div}^j u$, let us first take a look at the conventional transport equation. Given a velocity field $b : \mathbb{R}^{n+1} \to \mathbb{R}^n$ depending on (t, x), and initial data $\tau : \mathbb{R}^n \to \mathbb{R}$, this is written with unknown $I : \mathbb{R}^{n+1} \to \mathbb{R}$ as

$$\partial_t I + \langle \nabla_x I, b\rangle = 0, \quad I(0, \cdot\,) = \tau. \tag{2}$$

If b and τ are smooth, classical results on the ordinary differential equation $\gamma'(t) = b(t, \gamma(t))$ then show the existence of a unique smooth solution I. Starting with the renormalisation theory of DiPerna and Lions [3], a body of more recent research exists on relaxed assumptions that still ensure the meaningfulness and uniqueness of solutions to (2). Usually one, however, encounters an assumption of the type $\operatorname{div} b \in L^1(0, T; L^\infty(\mathbb{R}^n))$. This forces a great degree of regularity on the problem: as shown by Ambrosio [4], there still exists a "regular Lagrangian flow" that can transport $I(t, \cdot\,)$ between time instants. The least strict assumption that we have discovered is the one-sided Lipschitz condition (OSLC) of Bouchut et al. [5] that is, in fact, also a sufficient condition for uniqueness in Filippov's theory [6] on solutions to differential inclusions. Roughly speaking, it allows negative singularities or jumps in the distributional divergence of b, while disallowing positive ones. But we want them!

In the context of imaging, the differential equation of (2) is also known as the *optical flow constraint* or equation; see, e.g., Aubert and Kornprobst [7]. The vector field b describes the transformation of the scene $I(t, \cdot\,)$ at each time instant t into the one at following instants. In many imaging applications, the bounded-divergence theories are, however, insufficient. Consider a simple example of a ball thrown into the air, imaged from the side. (See Figure 1.) As the ball travels, part of the background becomes hidden, creating a *sink* or negative jump part in the distributional divergence of b. This situation is still covered by the OSLC. However, part of the scene is also revealed as the ball no longer occludes that part. There is a positive jump part in the divergence of b, or a *source*. This is no longer covered by the earlier studies. Our introduction of the term $\tau \operatorname{Div}^j u$ in (1) will, as we shall see, facilitate the modelling of this situation.

Our task then is to study properties of (1). We prove the continuity of a set-valued functional on (I, u) corresponding to (1), along with uniqueness and existence of solutions, subject to trace constraints. Throughout we assume I and u bounded in L^∞. While only convergence pointwise almost everywhere is required of I, much stronger form of convergence is required of u in our continuity results: a type of "segregated" weak convergence guaranteed by the SBD compactness theorem of Bellettini et al. [8] along with convergence of the total variations $|\operatorname{Div}^j u|(\mathbb{R}^{n+1})$. We show the existence of solutions to (1) subject to given traces in a rather weak distributional sense on the "source parts" $L_u^\pm$ of the jump set J_u. These are defined as where $\langle u^\pm, \pm\nu_{J_u}\rangle \geq 0$ and $\langle u^+ - u^-, \nu_{J_u}\rangle \neq 0$ (see Figure 2). The existence proof depends on approximating u by more regular functions. For this we refine the SBD approximation theorem of Chambolle [9, 10] to ensure the L^1 convergence of traces. As a byproduct, we are able to generalise the SBV approximation result of Cortesani and Toader [11] to the SBD case when $\mathcal{E}u \in L^2(\Omega)$, improving on an observation of Negri; see, e.g., [12, Proposition 2.4]. Finally, we provide a result on uniqueness of solutions to (1) subject traces on $L_u^\pm$. The proof is based on renormalisation arguments similar to DiPerna and Lions [3], and the related divergence chain rule due to Ambrosio et al. [13, 14].

Following the work of Borzì et al. [15], we will then apply condition (1) to an image interpolation problem. We want to fit to available data a space-time image $I \in \mathrm{BV}((0, T) \times \Omega)$ subject to the optical flow constraint. Employing SBD/BV regularisation, this problem is exemplified by

$$\min J(I,u) \quad \text{subject to (1) and } \|u\|_{L^\infty} \le M_u,\ \|I\|_{L^\infty} \le M_I \text{ with} \tag{3}$$

$$\begin{aligned} J(I,u) := \int_{\Omega_d} \|I - I_d\|_2^2 \, d\mathcal{L}^{n+1} + \theta |DI|(\mathbb{R}^{n+1}) \\ + \beta |E^j u|(\mathbb{R}^{n+1}) + \int \psi(|\mathcal{E}u|) \, d\mathcal{L}^{n+1} + \eta(\operatorname{Div}^j u) + \gamma \mathcal{H}^n(J_u). \end{aligned} \tag{4}$$

Here $\Omega_d \subset (0,T) \times \Omega$ is the domain where the data I_d is available. The term $\eta(\operatorname{Div}^j u)$ is a regularisation tool that we develop for ensuring the discussed convergence of $|\operatorname{Div}^j u|(\mathbb{R}^{n+1})$ subject to weak* convergence of $\operatorname{Div}^j u$. The latter is ensured by the other regularisation terms on u and the L^∞ bound.

When data is only available at initial and final times, solutions of (3) can be used in image registration applications. When more data is available, the solutions can be used for interpolation/reconstruction of video sequences, for example. In this imaging context, a considerable body of previous work on problems related to but different from (3) exists in literature. In addition to the already mentioned [15], we therefore restrict ourselves to pointing out just a few particular examples most directly related to our work through either a discontinuous setting or elastic, i.e., BD-type regularisation. Hinterberger et al. [16], for one, consider the problem of minimising $b \mapsto \int_\Omega \psi(|\partial_t I + \langle \nabla_x I, b\rangle|)$ at a *single* time instant when the image I and its space-time differential are known at that instant. These authors consider, among others, BD velocity fields, but expect considerable C^2 regularity from the known image. Aubert and Kornprobst [17], on the other hand, conduct an intricate study of a particular example case of this problem with the image also allowed to lie in SBV, while the velocity field is in BV with L^p divergence – a type of assumption seen in most work on the transport equation, as discussed above. Finally, in the paper of Keeling and Ring [18], the image registration problem of finding a space-time image I that satisfies given initial and final conditions is considered, minimising the deviation $\int \psi(|\partial_t I + \langle \nabla_x I, b\rangle|)$ from the optical flow constraint over all time instants. In this work also elastic regularisation is applied, but additional assumptions are made to ensure the velocity field lies in $H^1((0,T)\times\Omega)$.

The rest of this paper is arranged as follows. In Section 2 we introduce the basic notation and necessary preliminaries from the theory of functions of bounded deformation. In that section, we also prove the refined SBD approximation result. Then, in Section 3, we study the extension (1) of the transport equation (2). Finally, in Section 4 we briefly study theoretical properties of the optical flow fitting problem (3), and conclude the paper. The study of theoretical and numerical properties of discretisations of (1)–(4) is ongoing and future research.

2. Preliminaries

2.1. Basic notation

We denote the unit sphere in $\mathbb{R}^m$ by S^{m-1}, and the open ball of radius ρ centred at x by $B(x,\rho)$. The boundary of a set A is denoted ∂A, and the closure by $\operatorname{cl} A$. For $\nu \in \mathbb{R}^m$, we denote the orthogonal hyperplane by $\nu^\perp := \{z \in \mathbb{R}^m \mid \langle \nu, z\rangle = 0\}$.

The identity matrix is denoted id, and for $u, v \in \mathbb{R}^m$, we define $u \otimes v \in \mathbb{R}^{m\times m}$ by $(u\otimes v)(x) := u\langle v, x\rangle$. The trace of a matrix $A \in R^{m\times m}$ is denoted $\operatorname{Tr} A$, and the k-dimensional Jacobian of a linear map $L : \mathbb{R}^k \to \mathbb{R}^m$ ($k \le m$) is defined as $\mathcal{J}_k[L] := \sqrt{\det(L^* \circ L)}$.

We denote sets of functions essentially bounded by a given $M > 0$ by

$$L^\infty_M(A;B) := \{f : A \to B \mid \|f\|_{L^\infty(A;B)} \le M\}.$$

The space of (signed) finite Radon measures on an open set Ω is denoted $\mathcal{M}(\Omega)$. The k-dimensional Hausdorff measure, on any given ambient space $\mathbb{R}^m$, ($k \le m$), is denoted by $\mathcal{H}^k$, while $\mathcal{L}^m$ denotes the Lebesgue measure on $\mathbb{R}^m$. For a measure μ and a measurable set A, we denote by $\mu \llcorner A$ the measure defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$. The total variation measure of μ is denoted $|\mu|$. The upper and lower k-dimensional densities of a positive Radon measure μ at x are, respectively, defined as

$$\Theta^*_k(\mu, x) := \limsup_{\delta \searrow 0} \mu(B(x,\delta))/(\omega_k \delta^k), \quad \text{and} \quad \Theta_{*k}(\mu, x) := \liminf_{\delta \searrow 0} \mu(B(x,\delta))/(\omega_k \delta^k),$$

where ω_k is the volume of the unit ball in $\mathbb{R}^k$. When the limits agree, it is denoted Θ_k.

A set $\Sigma \subset \mathbb{R}^m$ is said to be *countably $\mathcal{H}^k$-rectifiable*, if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^m$, such that $\mathcal{H}^k(\Sigma \setminus \bigcup_{i=0}^\infty f_i(\mathbb{R}^k)) = 0$. If, moreover, $\mathcal{H}^k(\Sigma) < \infty$, then Σ is said to be *$\mathcal{H}^k$-rectifiable*.

If $\{A^i\}_{i=0}^\infty$ is a sequence of sets in a topological space X, we then define the outer and inner limits as

$$\limsup_{i\to\infty} A_i := \{x \in X \mid x^j \to x \text{ for some } x^j \in A^{i_j} \text{ and } 0 \le i_0 < i_1 < \ldots\}, \text{ and}$$
$$\liminf_{i\to\infty} A_i := \{x \in X \mid x^i \to x \text{ for some } x^i \in A^i \text{ with } i = 0, 1, 2, \ldots\}.$$

If $F : A \rightrightarrows B$ is a set-valued function between topological spaces A and B, it is said to be *outer-semicontinuous* if $\limsup_{i\to\infty} F(x^i) \subset F(x)$ for any $x^i \to x$, and *inner-semicontinuous* if $\liminf_{i\to\infty} F(x^i) \supset F(x)$ for any $x^i \to x$; see e.g. [19].

Finally, given a vector field $u \in L^\infty(\mathbb{R}^m; \mathbb{R}^m)$ such that the distributional divergence $\operatorname{Div} u$ is a Radon measure, we define the normal trace on an open set Ω with C^1 boundary as

$$\operatorname{Tr}(u, \partial\Omega)(\varphi) := \int_\Omega \langle \nabla\varphi, u\rangle \, d\mathcal{L}^m + \int_\Omega \varphi \, d \operatorname{Div} u, \quad (\varphi \in C^\infty_c(\mathbb{R}^m)).$$

The distribution $\operatorname{Tr}(u, \partial\Omega)$ is a function concentrated on $\partial\Omega$ and satisfying $\|\operatorname{Tr}(u, \partial\Omega)\|_{L^\infty(\partial\Omega;\mathbb{R}^m)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$; see [13]. Using this definition, one-sided normal traces $\operatorname{Tr}^\pm(u, \Sigma)$ can be defined on an oriented C^1 hypersurface Σ, and, by extension, oriented countably $\mathcal{H}^{m-1}$-rectifiable Σ.

2.2. Functions of bounded deformation

Following Temam [1], a function $u : \Omega \to \mathbb{R}^m$ on a bounded open set $\Omega \subset \mathbb{R}^m$, is said to be of *bounded deformation*, denoted $u \in \mathrm{BD}(\Omega)$, if its components are in $L^1(\Omega)$, and the symmetrised distributional gradient $Eu := (Du + (Du)^T)/2$ is a bounded measure. In other words, all the components $(D_i u_j + D_j u_i)/2$ with $i, j = 1, \ldots, m$ are measures with finite total variation.

If the boundary of Ω is Lipschitz (or C^1), then the trace $\operatorname{tr}(u, \partial\Omega)$ of u exists on $\partial\Omega$.

Similarly to functions of bounded variation (see, e.g., [20]), given a sequence $\{u^i\}_{i=1}^\infty \subset \mathrm{BD}(\Omega)$, strong convergence to $u \in \mathrm{BD}(\Omega)$ is defined as strong L^1 convergence $\|u^i - u\|_{L^1(\Omega)} \to 0$ together with convergence of the total variation $|E(u - u^i)|(\Omega) \to 0$. Weak convergence is defined as $u^i \to u$ strongly in $L^1(\Omega)$ along with $Eu^i \overset{*}{\rightharpoonup} Eu$ weakly* in $\mathcal{M}(\Omega)$.

According to Ambrosio et al. [2], the symmetrised gradient may be decomposed as $Eu = \mathcal{E}u\mathcal{L}^m + E^j u + E^c u$, where $\mathcal{E}u$ is the density of the absolutely continuous part, and equals $(\nabla u + (\nabla u)^T)/2$ $\mathcal{L}^m$-a.e. We sometimes use the notation $E^a u := \mathcal{E}u\mathcal{L}^m$. The *jump part* $E^j u$ may be represented as

$$\begin{aligned} E^j u &= (u^+ - u^-) \odot \nu_{J_u} \mathcal{H}^{m-1} \llcorner J_u \\ &:= \frac{1}{2}\bigl((u^+ - u^-) \otimes \nu_{J_u} + \nu_{J_u} \otimes (u^+ - u^-)\bigr)\mathcal{H}^{m-1} \llcorner J_u, \end{aligned} \tag{5}$$

where x is in the *jump set* J_u of u if for some $\nu := \nu_{J_u}(x)$ there exist one-sided traces $u^\pm(x)$ defined as satisfying

$$\lim_{\rho \searrow 0} \frac{1}{\rho^m} \int_{B^\pm(x,\rho,\nu)} \|u^\pm(x) - u(y)\| \, dy = 0, \tag{6}$$

where $B^\pm(x,\rho,\nu) := \{y \in B(x,\rho) \mid \pm\langle y - x, \nu\rangle \geq 0\}$. It turns out that J_u is countably $\mathcal{H}^{m-1}$-rectifiable, and ν is (a.e.) the normal to J_u. The remaining *Cantor part* $E^c u$ vanishes on any Borel set σ-finite with respect to $\mathcal{H}^{m-1}$. The space $\mathrm{SBD}(\Omega)$ of *special functions of bounded deformation* is defined as those $u \in \mathrm{BD}(\Omega)$ with $E^c u = 0$.

We may write the distributional divergence of u as $\operatorname{Div} u = \sum_{i=1}^m \langle \xi_i, Eu\xi_i\rangle = \operatorname{Tr} Eu$ when $\xi_1, \ldots, \xi_m$ is the standard basis of $\mathbb{R}^m$. Accordingly, the absolutely continuous part, $\operatorname{div} u$, can be defined through $\operatorname{div} u = \sum_{i=1}^m \langle \xi_i, \mathcal{E}u\xi_i\rangle$, while the jump part of the divergence is defined as $\operatorname{Div}^j u := \sum_{i=1}^m \langle \xi_i, E^j u\xi_i\rangle$. This may also be written

$$\operatorname{Div}^j u = \langle u^+ - u^-, \nu_{J_u}\rangle \mathcal{H}^{m-1} \llcorner J_u.$$

We denote by S_u the complement of the set where the Lebesgue limit $\widetilde{u}$ exists. The latter is, of course, defined by

$$\lim_{\rho \searrow 0} \frac{1}{\rho^m} \int_{B(x,\rho)} \|\widetilde{u}(x) - u(y)\| \, dy = 0.$$

Finally, we will be employing one-dimensional slices (or sections) of functions $u \in \mathrm{BD}(\Omega)$. These are defined by $u^{[y,\xi]}(t) := \langle u(y + t\xi), \xi\rangle$ for $y, \xi \in \mathbb{R}^m$. We also let

$$\Omega^{[\xi]} := \{y \in \xi^\perp \mid y + t\xi \in \Omega \text{ for some } t \in \mathbb{R}\}, \quad \text{and} \quad \Omega^{[y,\xi]} := \{t \in \mathbb{R} \mid y + t\xi \in \Omega\}.$$

For the jump set J_u, we set $J_{u,\xi} := \{x \in J_u \mid \langle u^+(x) - u^-(x), \xi\rangle \neq 0\}$. Then the Structure Theorem of Ambrosio et al. [2] can be stated.

Theorem 1 (Structure Theorem [2]). *Suppose $u \in \mathrm{BD}(\Omega)$, and $\xi \in \mathbb{R}^m \setminus \{0\}$. Then the following points hold.*

1. *For any Borel set $A \subset \Omega$, we have $\langle \xi, E^a u\xi\rangle(A) = \int_{\Omega^{[\xi]}} D^a u^{[y,\xi]}(A^{[y,\xi]}) \, d\mathcal{H}^{m-1}(y)$ and $|\langle \xi, E^a u\xi\rangle|(A) = \int_{\Omega^{[\xi]}} |D^a u^{[y,\xi]}|(A^{[y,\xi]}) \, d\mathcal{H}^{m-1}(y)$.*
2. *For any Borel set $A \subset \Omega$, we have $\langle \xi, E^j u\xi\rangle(A) = \int_{\Omega^{[\xi]}} D^j u^{[y,\xi]}(A^{[y,\xi]}) \, d\mathcal{H}^{m-1}(y)$ and $|\langle \xi, E^j u\xi\rangle|(A) = \int_{\Omega^{[\xi]}} |D^j u^{[y,\xi]}|(A^{[y,\xi]}) \, d\mathcal{H}^{m-1}(y)$.*
3. *The slices $u^{[y,\xi]}, \widetilde{u}^{[y,\xi]} \in \mathrm{BV}(\Omega^{[y,\xi]})$ with $u^{[y,\xi]} = \widetilde{u}^{[y,\xi]}$ a.e. with respect to $\mathcal{L}^1 \llcorner \Omega^{[y,\xi]}$.*
4. *For $\mathcal{H}^{m-1}$-a.e. $y \in \Omega^{[\xi]}$, the jump sets satisfy $J_{u^{[y,\xi]}} = J_{u,\xi}^{[y,\xi]}$, and we have*

$$\langle \xi, u^\pm(y + t\xi)\rangle = (u^{[y,\xi]})^\pm(t) = \lim_{s \to t^\pm} \widetilde{u}^{[y,\xi]}(s)$$

for every $t \in J_{u,\xi}^{[y,\xi]}$. The normals of J_u and $J_{u^{[y,\xi]}}$ are oriented to satisfy $\langle \nu_{J_u}, \xi\rangle \geq 0$ when $\nu_{J_u^{[y,\xi]}} = 1$.

Here $D^a f$ and $D^j f$ denote the absolutely continuous and jump parts of the distributional gradient Df of a function f of bounded variation. In the present one-dimensional setting of $f : \Omega^{[y,\xi]} \subset \to \mathbb{R}$, they are equal to $E^a f$ and $E^j f$, but see [2, 20] for details.

The following compactness result of Bellettini et al. [8] will also be important.

Theorem 2 (SBD compactness [8])**.** *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$. If $\{u^i\}_{i=0}^\infty \subset \mathrm{SBD}(\Omega)$ with*

$$\|u^i\|_{L^1} + \int_\Omega \psi(|\mathcal{E}u^i|)\,dx + |E^j u^i|(\Omega) + \mathcal{H}^{m-1}(J_{u^i}) \leq K < \infty,$$

then there exists a subsequence of $\{u^i\}_{i=0}^\infty$, unrelabelled, such that

$$u^i \to u \text{ strongly in } L^1(\Omega), \tag{7}$$

$$\mathcal{E}u^i \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega), \tag{8}$$

$$E^j u^i \overset{*}{\rightharpoonup} E^j u \text{ weakly* in } \mathcal{M}(\Omega), \text{ and} \tag{9}$$

$$\mathcal{H}^{m-1}(J_u) \leq \liminf \mathcal{H}^{m-1}(J_{u^i}). \tag{10}$$

2.3. An approximation result

In the following Theorem 3 we provide a refinement of the SBD approximation theorem of Chambolle [9, 10]. Under the additional condition that u is essentially bounded, our claim is the L^1 convergence of one-sided traces on the jump set. In fact, we find (see [1]) that traces in general are convergent due to the consequent strong convergence of the approximations.

Definition. Given an open set $\Omega \subset \mathbb{R}^m$, we denote by $\mathcal{W}^\infty(\Omega)$ the set of functions $u : \Omega \to \mathbb{R}^m$ that are in $C^\infty(\Omega \setminus \mathrm{cl}\, J)$ for some essentially closed $J \subset \Omega$ (i.e. $\mathcal{H}^{m-1}((\mathrm{cl}\, J \cap \Omega) \setminus J) = 0$) that is contained in the union of finitely many closed connected pieces of C^1 surfaces (of dimension $m-1$).

Definition. We say that a bounded open set $\Omega \subset \mathbb{R}^m$ has C^0 boundary $\partial\Omega$ if at each $x \in \partial\Omega$, there exists a neighbourhood U of x, a unit vector $e \in \mathbb{R}^m$ and a continuous map $f : e^\perp \to \mathbb{R}$, such that $U \cap \partial\Omega = U \cap \{x + f(x)e \mid x \in e^\perp\}$.

Theorem 3. *Let $\Omega \subset \mathbb{R}^m$ be an open bounded set with C^0 boundary $\partial\Omega$. Suppose $u \in \mathrm{SBD}(\Omega) \cap L^\infty_M(\Omega; \mathbb{R}^m)$ for some $M < \infty$, and that u satisfies the bound*

$$P(u) := \int_\Omega W(\mathcal{E}u(x))\,dx + \mathcal{H}^{m-1}(J_u) < \infty, \quad \text{where } W(A) := \mathrm{Tr}(AA^T) + (\mathrm{Tr}(A))^2/2.$$

Then there exists a sequence $\{u^i\}_{i=0}^\infty \subset \mathcal{W}^\infty(\Omega) \cap L^\infty_M(\Omega; \mathbb{R}^m)$ satisfying

$$u^i \to u \text{ strongly in } L^2(\Omega; \mathbb{R}^m), \tag{11}$$

$$\mathcal{E}u^i \to \mathcal{E}u \text{ strongly in } L^2(\Omega; \mathbb{R}^{m\times m}), \tag{12}$$

$$\int_{J_u \cup J_{u^i}} \big\|(u^i)^\pm(x) - u^\pm(x)\big\|\, d\mathcal{H}^{m-1}(x) \to 0, \text{ and} \tag{13}$$

$$\mathcal{H}^{m-1}(J_{u^i} \Delta J_u) \to 0. \tag{14}$$

In particular $|Eu^i - Eu|(\Omega) \to 0$, so $\{u^i\}_{i=0}^\infty$ converge to u strongly in $\mathrm{BD}(\Omega)$.

Proof. The proof and the construction are essentially the same as those presented in [9], with some additional observations and minor modifications. As the proof is long, we will therefore not attempt to replicate it in full. Rather, we sketch the overall idea of the original proof for the reader's convenience, and then describe the modifications and additional observations needed.

Given $\epsilon > 0$, first in [9, Theorem 2] a Besicovitch covering argument is used on J_u. This yields a finite collection of open balls B_j, $(j = 1, \ldots, k)$, such that the corresponding closed balls $\operatorname{cl} B_j$ are mutually disjoint, $\mathcal{H}^{m-1}(J_u \cap \partial B_j) = 0$, and $\mathcal{H}^{m-1}(J_u \setminus \bigcup_{j=1}^k B_j) < \epsilon$. Moreover, minding that J_u is $\mathcal{H}^n$-rectifiable and hence contained on at most countably many C^1 surfaces $\{\Gamma_\ell\}_{\ell=0}^\infty$, the balls B_j are asked to be split into two open halves $U_j^\pm$ by some Γ_{ℓ_j}, and to satisfy

$$\mathcal{H}^{m-1}((\Gamma_{\ell_j} \Delta J_u) \cap \operatorname{cl} B_j) \leq \epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j). \tag{15}$$

Next, it is set $A_t := \{x \in \mathbb{R}^m \mid \operatorname{dist}(x, \Omega \setminus \bigcup_{j=1}^k \operatorname{cl} B_j) < t\}$ for some small enough $t > 0$ that $\mathcal{H}^m(J_u \cap A_t) \leq 2\epsilon$. Then a sequence of approximations $\{u_U^i\}_{i=0}^\infty \subset \mathcal{W}^\infty(U) \cap L_M^\infty(U; \mathbb{R}^m)$ is constructed separately on each $U = U_1^\pm, \ldots, U_k^\pm, (A_t \cap \Omega)$ by invoking Lemma 2 below, refining [9, Theorem 1]. The approximations satisfy for some constant $c_m > 0$ that

$$\begin{aligned} &\|u_U^i - u\|_{L^2(U;\mathbb{R}^m)} \to 0, \\ &\|\operatorname{tr}(u_U^i, \partial U) - \operatorname{tr}(u, \partial U)\|_{L^1(\partial U;\mathbb{R}^m)} \to 0, \text{ and} \end{aligned} \tag{16}$$

$$\limsup_{i\to\infty} \int_U W(\mathcal{E}u_U^i(x))\,dx + \mathcal{H}^{m-1}(\operatorname{cl}(J_{u_U^i} \cap U)) \leq \int_U W(\mathcal{E}u(x))\,dx + c_m\mathcal{H}^{m-1}(J_u \cap U). \tag{17}$$

Setting $u_{B_j}^i(x) = u_{U_j^\pm}^i(x)$ when $x \in U_j^\pm$, the approximations $u_{B_1}^i, \ldots, u_{B_k}^i, u_{A_t}^i$ are then combined for large enough i (see [9, Lemma 3.1]) using a partition of unity on $B_1, \ldots, B_k, A_t$ to yield a final approximation u_ϵ with energy $P(u_\epsilon)$ that does not exceed $P(u)$ by more than a constant factor of ϵ. Defining $u^i := u_{\epsilon^i}$ for a sequence $\epsilon^i \searrow 0$, the claims (11), (12), and

$$\mathcal{H}^{m-1}(J_{u^i}) \to \mathcal{H}^{m-1}(J_u) \tag{18}$$

of the original approximation result now follow without much effort from a variant of Theorem 2; see [9, Theorem 3].

We now have to prove (13) and (14). Let us observe that thanks to (16) we have

$$R_j^i := \int_{J_u \cap B_j \cap \Gamma_{\ell_j}} \|(u_{B_j}^i)^\pm(x) - u^\pm(x)\|\,d\mathcal{H}^{m-1}(x) \to 0, \quad (i \to \infty).$$

Minding that $J_{u_{B_j}^i}$ consists of points $x \in B_j$ such that there exists two different one-sided limits $(u_{B_j}^i)^+(x) \neq (u_{B_j}^i)^-(x)\}$, it follows that also

$$H_j^i := \mathcal{H}^{m-1}(J_u \cap B_j \cap \Gamma_{\ell_j} \setminus J_{u_{B_j}^i}) \to 0, \quad (i \to \infty).$$

For the proof of this fact we refer to Lemma 7 in the Appendix. (There we take $A = J_u \cap B_j \cap \Gamma_{\ell_j}$, $v^i = (u_{B_j}^i)^+ - (u_{B_j}^i)^-$, and $\mu = \mathcal{H}^{m-1}$.) Hence we may deduce that if we take i_j^ϵ large enough, then both

$$R_j^i,\ MH_j^i \leq M\epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j), \quad (i \geq i_j^\epsilon).$$

From (15) we also have

$$\mathcal{H}^{m-1}(J_u \cap B_j \setminus \Gamma_{\ell_j}) \le \epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j).$$

Minding that $\|u\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M$, we then get the estimate

$$\begin{aligned}\int_{J_u \cap B_j} \big\|(u^i_{B_j})^\pm(x) - u^\pm(x)\big\| \, d\mathcal{H}^{m-1}(x) &\le R^i_j + M\mathcal{H}^{m-1}(J_u \cap B_j \setminus \Gamma_{\ell_j}) \\ &\le 2M\epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j), \quad (i \ge i^\epsilon_j),\end{aligned} \tag{19}$$

along with

$$\begin{aligned}\mathcal{H}^{m-1}(J_u \cap B_j \setminus J_{u^i_{B_j}}) &\le H^i_j + \mathcal{H}^{m-1}(J_u \cap B_j \setminus \Gamma_{\ell_j}) \\ &\le 2\epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j), \quad (i \ge i^\epsilon_j).\end{aligned} \tag{20}$$

Since the balls B_j are mutually disjoint, constructing u_ϵ with $i \ge i^\epsilon_j$, we therefore have by summing over the estimates (19) on $B_1, \ldots, B_k$ and the bound $\mathcal{H}^m(J_u \cap A_t) \le 2\epsilon$ on A_t that

$$\begin{aligned}\int_{J_u} \big\|u^\pm_\epsilon(x) - u^\pm(x)\big\| \, d\mathcal{H}^{m-1}(x) &\le 2\epsilon M + \sum_{j=1}^k \big(2M\epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j)\big) \\ &\le 2M\epsilon\big(1 + \mathcal{H}^{m-1}(J_u)/(1-\epsilon)\big).\end{aligned} \tag{21}$$

Likewise, employing (20), we deduce that

$$\mathcal{H}^{m-1}(J_u \setminus J_{u_\epsilon}) \le 2\epsilon + \sum_{j=1}^k \big(2\epsilon/(1-\epsilon)\mathcal{H}^{m-1}(J_u \cap B_j)\big) \le 2\epsilon(1 + \mathcal{H}^{m-1}(J_u)/(1-\epsilon)). \tag{22}$$

Recalling that $u^i := u_{\epsilon i}$ and combining (22) with (18), we obtain (14). In particular, $\mathcal{H}^{m-1}(J_{u^i} \setminus J_u) \to 0$. Employing the bound $\|u\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M$, this implies

$$\int_{J_{u^i} \setminus J_u} \big\|(u^i)^\pm(x) - u(x)\big\| \, d\mathcal{H}^{m-1}(x) \to 0, \quad (i \to \infty).$$

Combining this observation with (21) completes the proof of (13).

We must still show strong convergence. Thanks to $\mathcal{L}^m(\Omega) < \infty$, it follows that the L^2 convergences (11) and (12) hold in L^1 as well. Thus, in particular, $\|\mathcal{E}u^i - \mathcal{E}u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$. From (13) it follows that $|E^j u^i - E^j u|(\Omega) \to 0$. Combined, we find $|Eu^i - Eu|(\Omega) \to 0$, so the claimed strong convergence follows. □

To prove Lemma 2 employed in the above proof, we first need the following extension result.

Lemma 1. *Suppose $\Omega \subset \mathbb{R}^m$ is a bounded open set with C^0 boundary $\partial\Omega$. Let $u \in \mathrm{SBD}(\Omega) \cap L^\infty_M(\Omega;\mathbb{R}^m)$ be given with $P(u) < \infty$. Then for any $\epsilon > 0$ there exists $\Omega' \Supset \Omega$ and $u' \in \mathrm{SBD}(\Omega') \cap L^\infty_M(\Omega';\mathbb{R}^m)$ with*

$$\big\|u' - u\big\|_{L^2(\Omega;\mathbb{R}^m)} \le \epsilon, \tag{23}$$

$$\int_{\Omega'} W(\mathcal{E}u'(x)) \, dx \le \int_\Omega W(\mathcal{E}u(x)) \, dx + \epsilon, \tag{24}$$

$$\mathcal{H}^{m-1}(J_{u'}) \le \mathcal{H}^{m-1}(J_u) + \epsilon, \text{ and} \tag{25}$$

$$\big\|\mathrm{tr}(u', \partial\Omega) - \mathrm{tr}(u, \partial\Omega)\big\|_{L^1(\partial\Omega;\mathbb{R}^m)} \le \epsilon. \tag{26}$$

Moreover, $\mathcal{H}^{m-1}$-a.e. point $x \in \partial\Omega$ is a Lebesgue point of u'.

Proof. This lemma improves [9, Lemma 3.2], and the construction employed is nearly the same, just with more meticulous choice of the perturbations z_i^t, where originally simply $z_i^t = te_i$. We will therefore not prove (23)–(25) as they follow exactly as in [9]. We only describe the construction employed and show (26) together with the Lebesgue point property.

The construction is as follows. Thanks to $\partial\Omega$ being of class C^0, we may cover it with finitely many open balls $\{A_i\}_{i=1}^k$, such that there is another set of open balls $B_i \Supset A_i$, directions $e_i \in S^{m-1}$, and continuous maps $f_i : D_i \subset e_i^\perp \to \mathbb{R}$ that give $B_i \cap \partial\Omega = B_i \cap \{x + f_i(x)e_i \mid x \in D_i\}$. In fact, we may assume f_i uniformly continuous, since we may replace B_i by a smaller ball containing A_i. For $t > 0$, let us set

$$Z_i^t := \bigl\{z \in \mathbb{R}^m \bigm| \bigl\|z - \langle z, e_i\rangle e_i\bigr\| \leq \langle z, e_i\rangle \in (0, t],\ A_i \cap \operatorname{cl}\Omega \subset A_i \cap (\Omega + z)\bigr\}.$$

That is, Z_i^t is the subset of perturbations in a truncated cone with axis e_i that satisfy $A_i \cap \operatorname{cl}\Omega \subset A_i \cap (\Omega + z)$. For small perturbations z such that $A_i \subset B_i + z$, this latter condition may equivalently be written as

$$f_i(x) < f_i(x - (z - \langle z, e_i\rangle e_i)) + \langle z, e_i\rangle \text{ for } x \in D_i \text{ with } x + f_i(x)e_i \in A_i.$$

If we define the slices $\bar{Z}_i^s := Z_i^s \cap \{z \in \mathbb{R}^m \mid \langle z, e_i\rangle = s\}$, $(s > 0)$, then $z \in \bar{Z}_i^s$ follows if both $A_i \subset B_i + z$ and

$$f_i(x) - f^i(x + se_i - z) < s \text{ for } x \in D_i \text{ with } x + f_i(x)e_i \in A_i.$$

Minding the inclusion $B_i \Supset A_i$ and the uniform continuity of f_i, it follows that for each $s > 0$ there exists $\delta_s > 0$ such that $z \in \bar{Z}_i^s$ if $\langle z, e_i\rangle = s$ and $\|z - se_i\| < \delta_s$. Hence $\mathcal{H}^{m-1}(\bar{Z}_i^s) > 0$, so that also

$$\mathcal{L}^m(Z_i^t) = \int_0^t \mathcal{H}^{m-1}(\bar{Z}_i^s)\, ds > 0.$$

For each $t > 0$, let us now choose some $z_i^t \in Z_i^t$, to be determined later in more detail. Observe that $z_i^t \to 0$ as $t \searrow 0$. Within $A_i \cap (\Omega + z_i^t)$, we then define $u_i^t(x) := u(x - z_i^t)$. We also choose $A_0 \Subset \Omega$ such that $\operatorname{cl}\Omega \subset \bigcup_{i=0}^k A_i$, and set $u_0^t(x) := u(x)$ in A_0. We choose a special smooth partition of unity $\varphi_0, \ldots, \varphi_k$ on $A_0, \ldots, A_k$, given by [9, Lemma 3.1], that satisfies $\mathcal{H}^{m-1}((J_u \cap (\bigcup_{i=0}^k \operatorname{supp}\operatorname{cl}\{0 < \varphi_i < 1\})) \leq \epsilon/(2(k+1))$. Then, we let $u^t := \sum_{i=0}^k \varphi_i u_i^t$, which is a function in $\mathrm{SBD}(\Omega^t)$ for $\Omega^t := A_0 \cup \bigcup_{i=1}^k (A_i \cap (\Omega + z_i^t))$.

The properties (23)–(25) now hold for $u' = u^t$ and $\Omega' = \Omega_t$ when t is small enough, exactly as shown in [9]. To show (26), we first observe that $E^j u(\cdot - z_i^t) \overset{*}{\rightharpoonup} E^j u$ as well as $\bigl|E^j u\bigr|(\cdot - z_i^t) \overset{*}{\rightharpoonup} \bigl|E^j u\bigr|$ weakly* as measures as $t \searrow 0$. Secondly, from the expression (5) for $E^j u$, and the continuity of φ_i, we observe that $E^j(\varphi_i u_i^t) = \varphi_i E^j u_i^t$. Therefore, for any $\varphi \in C_c(\mathbb{R}^m)$,

$$\begin{aligned} E^j(\varphi_i u_i^t)(\varphi) &= (\varphi_i E^j u_i^t)(\varphi) = E^j u(\cdot\, - z_i^t)(\varphi_i \varphi) \\ &\to E^j u(\varphi_i \varphi) = (\varphi_i E^j u)(\varphi) = E^j(\varphi_i u)(\varphi), \quad (t \searrow 0; i = 0, \ldots, k), \end{aligned}$$

so $E^j(\varphi_i u_i^t) \overset{*}{\rightharpoonup} \varphi_i E^j u$ weakly* in $\mathcal{M}(\mathbb{R}^m)$. Consequently, weakly* in $\mathcal{M}(\mathbb{R}^m)$, we have

$$E^j u^t = \sum_{i=0}^k E^j(\varphi_i u_i^t) \overset{*}{\rightharpoonup} \sum_{i=0}^k \varphi_i E^j u = E^j u, \quad (t \searrow 0). \tag{27}$$

Likewise, minding that $\varphi \geq 0$, we find that $\left|E^j(\varphi_i u_i^t)\right| \overset{*}{\rightharpoonup} \varphi_i\left|E^j u\right|$ as $t \searrow 0$ for $i = 0, \ldots, k$. Now, $\left|E^j u^t\right|(\Omega) \to \left|E^j u\right|(\Omega)$ as $t \searrow 0$ follows similarly to (27) if we can show that the total variations measures decompose as $\left|\sum_{i=0}^k \varphi_i E^j u_i^t\right| = \sum_{i=0}^k \left|\varphi_i E^j u_i^t\right|$. Towards this end, we have to choose the perturbations $z_i^t \in Z_i^t$ carefully. By application of Fubini's theorem, we observe that the set

$$N := \{z \in \mathbb{R}^m \mid \mathcal{H}^{m-1}(J_u \cap (z + J_u)) > 0\}$$

has zero $\mathcal{L}^m$ measure, as do

$$N_i := \{z \in \mathbb{R}^m \mid \mathcal{H}^{m-1}((J_u + z) \cap \partial\Omega \cap A_i) > 0\}, \quad (i = 1, \ldots, k).$$

Therefore, since $\mathcal{L}^m(Z_i^t) > 0$ for $t > 0$, as we have shown, it is possible to make the choices

$$z_i^t \in Z_i^t \setminus (N_i \cup N \cup \bigcup_{j<i} (N + z_t^j)).$$

Since $J_{u_i^t} = (J_u + z_i^t) \cap (A_i \cap \Omega^t)$, we then find that $\mathcal{H}^{m-1}(J_{u_i^t} \cap J_{u_j^t} \cap (A_i \cap A_j \cap \Omega^t)) = 0$ for all $i \neq j$ with $i, j = 0, 1, 2, \ldots, k$. (The sets N_i have not been employed yet; we will use them shortly to get the claim on the Lebesgue points.) Clearly, minding (5), we now have $\left|\sum_{i=1}^k \varphi_i E^j u_i^t\right| = \sum_{i=0}^k \left|\varphi_i E^j u_i^t\right|$. Thus $\left|E^j u^t\right|(\Omega) \to \left|E^j u\right|(\Omega)$ as $t \searrow 0$.

Now, observe that $\mathcal{E}u^t \to \mathcal{E}u$ strongly in $L^1(\Omega; \mathbb{R}^{m \times m})$ due to the strong convergence in $L^2(\bigcup_{i=0}^k A_i; \mathbb{R}^{m \times m})$ shown in [9], and $\mathcal{L}^m(\Omega) < \infty$. It follows that $\left|Eu^t\right|(\Omega) \to \left|Eu\right|(\Omega)$ as $t \searrow 0$. Knowing (23), also $u^t \to u$ strongly in $L^1(\Omega; \mathbb{R}^m)$. Hence we find that u^t converges to u "in the intermediate sense" on Ω. But the trace operator into L^1 is continuous in the topology of intermediate convergence by [1, Theorem 3.1]. This gives (26) for $u' = u^t$ and $\Omega' = \Omega^t$ when t is small enough.

Finally, to show that $\mathcal{H}^{m-1}$-a.e. point $x \in \partial\Omega$ is a Lebesgue point of u', first observe that for $i = 1, \ldots, k$, we have $\mathcal{H}^{m-1}((J_u + z_i^t) \cap (\partial\Omega \cap A_i)) = 0$ due to $z_i^t \notin N_i$. This gives

$$\mathcal{H}^{m-1}(J_{u_i^t} \cap (\partial\Omega \cap A_i)) = 0.$$

But, recalling that $S_{u_i^t}$ denotes the complement of the Lebesgue set of u_i^t, we also have

$$\mathcal{H}^{m-1}(S_{u_i^t} \setminus J_{u_i^t} \cap (\partial\Omega \cap A_i)) = 0.$$

This follows by choosing $v = c\chi_{\Omega \cap A_i} \in \mathrm{BD}(A_i)$ for some constant $c \in \mathbb{R}^m \setminus \{0\}$ in [2, Theorem 6.1], which claims that $|Ev|(S_{u_i^t} \setminus J_{u_i^t}) = 0$ for any $v \in \mathrm{BD}(A_i)$. Minding that $A_0 \Subset \Omega$, we have therefore shown that

$$\mathcal{H}^{m-1}(S_{u_i^t} \cap (\partial\Omega \cap A_i)) = 0, \quad (i = 0, \ldots, k).$$

But this says that $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega \cap A_i$ is a Lebesgue point of u_i^t. Hence, as $u^t = \sum_{i=0}^k \varphi_i u_i^t$ and the partition of unity φ_i is smooth, we observe as claimed that $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$ is a Lebesgue point of u^t. $\square$

Lemma 2. *Suppose $\Omega \subset \mathbb{R}^m$ is a bounded open set with C^0 boundary $\partial\Omega$. Let $u \in \mathrm{SBD}(\Omega) \cap L^\infty_M(\Omega; \mathbb{R}^m)$ be given with $P(u) < \infty$. Then there exists a sequence $\{u^i\}_{i=0}^\infty \subset \mathcal{W}^\infty(\Omega) \cap L^\infty_M(\Omega; \mathbb{R}^m)$*

with each J_{u^i} contained on finitely many $(m-1)$-simplices, and we have

$$\|u^i - u\|_{L^2(\Omega;\mathbb{R}^m)} \to 0, \tag{28}$$

$$\|\mathrm{tr}(u^i, \partial\Omega) - \mathrm{tr}(u, \partial\Omega)\|_{L^1(\partial\Omega;\mathbb{R}^m)} \to 0, \text{ and} \tag{29}$$

$$\limsup_{i\to\infty} \int_\Omega W(\mathcal{E}u^i(x))\,dx + \mathcal{H}^{m-1}(\mathrm{cl}\, J_{u^i}) \le \int_\Omega W(\mathcal{E}u(x))\,dx + c_m\mathcal{H}^{m-1}(J_u), \tag{30}$$

where c_m is a constant depending on the dimension m only.

Proof. Once again, the construction and proof of this lemma are a refinement of [9, Theorem 1], so we will only describe the overall idea and the additions needed to achieve our claims.

The first step of the construction is to choose an arbitrary $\epsilon > 0$ and apply Lemma 1 to extend u as u' from Ω onto a larger set $\Omega' \Supset \Omega$. Then a finite element approximation of u' is performed on Ω', while also preventing the blow-up of $\mathcal{E}u^i$ and approximating the jump set of u' with "jump cubes". The selection of these jump cubes and the interpolation grid is a rather lengthy process, but for our purposes it suffices to mention that it is possible to choose arbitrarily a shift y from a subset of positive measure of $[0,1)^m$, such that when v^h for $h > 0$ is constructed as described next, then

$$\|v^h - u'\|_{L^2(\Omega;\mathbb{R}^m)} \to 0, \quad (h \searrow 0), \quad \text{and} \tag{31}$$

$$\int_\Omega W(\mathcal{E}v^h(x))\,dx + \mathcal{H}^{m-1}(\mathrm{cl}\, J_{v^h}) \le \int_{\Omega'} W(\mathcal{E}u'(x))\,dx + c_m\mathcal{H}^{m-1}(J_{u'}). \tag{32}$$

To proceed with the construction, let us choose an orthonormal basis $\{e_j\}_{j=1}^m$ of $\mathbb{R}^m$, satisfying $\mathcal{H}^{m-1}(\{x \in J_u \mid \langle (u')^+(x) - (u')^-(x), e\rangle = 0\}) = 0$ for all $e \in \{e_k\}_{k=1}^m \cup \{e_k - e_\ell\}_{k,\ell=1}^m$. Then, given $h > 0$ and letting $G := \sum_{j=1}^m \mathbb{Z}e_j$, finite element interpolation is performed on the grid $hy + hG \cap \Omega'$ with shape functions of the form $\Delta(x) := \prod_{j=1}^m \max\{0, 1 - |\langle e_j, x\rangle|\}$, to yield

$$w^h(x) := \sum_{\xi \in hG\cap\Omega'} u'(hy + \xi)\Delta((x - \xi)/h - y).$$

Next, cubes $\xi + hQ_y$ for $\xi \in h\mathbb{Z}^m$ and $Q_y := y + \sum_{j=1}^m [0,1)e_j$ are chosen as jump cubes if $\xi \in J_{u'} + hV$, where V is a one-dimensional set modelling the interactions between different nodes ξ. Then the original final approximation in [9], satisfying (31), (32), is obtained by setting $v^h(x)$ to $w^h(x)$ whenever x does not belong to a jump cube, and $v^h(x)$ to 0 when x does belong to a jump cube. We will have to alter this construction a bit on the jump cubes.

In the original proof, the shift $y \in [0,1)^m$ is chosen arbitrarily from a subset of eligible points of positive $\mathcal{L}^m$-measure. We can therefore assume that the choice is such that all the points of $hy + (hG \cap \Omega')$, used in the construction of w^h, are Lebesgue points of u'. Since $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$ is, by Lemma 1, likewise a Lebesgue point of u', it therefore follows from a simple mollification argument that w^h is convergent pointwise a.e. to u' on $\partial\Omega$. Since u' is bounded and $\mathcal{H}^{m-1}(\partial\Omega) < \infty$, the Egorov and Vitali convergence theorems then establish $L^1(\partial\Omega;\mathbb{R}^m)$ convergence of the traces $\mathrm{tr}(w^h, \partial\Omega)$ to $\mathrm{tr}(u', \partial\Omega)$. But the convergence of traces may not hold for v^h, as $\partial\Omega$ may be covered by jump cubes. We therefore modify the construction as follows. Again on the grid $hy + hG \cap \Omega'$, we define the piecewise constant approximations

$$\bar{w}^h(x) := \sum_{\xi \in hG\cap\Omega'} \frac{\chi_{(\xi+hQ_y)\cap\Omega'}(x)}{\mathcal{L}^m((\xi + hQ_y) \cap \Omega')} \int_{(\xi+hQ_y)\cap\Omega'} u'(x)\,dx.$$

As above, we then observe that the traces $\operatorname{tr}(\bar{w}^h, \partial\Omega)$ converge to $\operatorname{tr}(u', \partial\Omega)$ in $L^1(\partial\Omega; \mathbb{R}^m)$. Also $\bar{w}^h \to u'$ strongly in $L^2(\Omega'; \mathbb{R}^m)$ due to standard approximation results. Now we set

$$\bar{v}^h(x) := \begin{cases} w^h(x), & x \text{ belongs to a jump cube,} \\ \bar{w}^h(x), & x \text{ does not belong to a jump cube.} \end{cases}$$

By the discussion above and (31), it easily follows that

$$\left\|\bar{v}^h - u'\right\|_{L^2(\Omega;\mathbb{R}^m)} \to 0, \quad \text{and} \quad \left\|\operatorname{tr}(\bar{v}^h, \partial\Omega) - \operatorname{tr}(u', \partial\Omega)\right\|_{L^1(\partial\Omega;\mathbb{R}^m)} \to 0, \quad (h \searrow 0). \tag{33}$$

Regarding (32), we observe that this modification does not alter the energies $\int_\Omega W(\mathcal{E}v^h(x))\,dx$, the function $\bar{w}^h$ being constant on each jump cube. Moreover, the estimate (32) was actually obtained in [9] through the estimates

$$\begin{gathered} \operatorname{cl} J_{v^h} \subset \bigcup \{\partial(\xi + hQ_y) \mid \xi \in hG,\ \xi + hQ_y \text{ jump cube}\}, \qquad \text{and} \\ \int_\Omega W(\mathcal{E}v^h(x))\,dx + K_h \mathcal{H}^{m-1}(hQ_y) \leq \int_{\Omega'} W(\mathcal{E}u'(x))\,dx + c_m \mathcal{H}^{m-1}(J_{u'}), \end{gathered}$$

where K_h the number of jump cubes. But the jump cubes are not changed by our altered construction (although the jump set J_{v^h} contained on their boundaries may be), so (32) continues to hold for $\bar{v}^h$.

One issue however remains. The approximations $\bar{v}^h$ are Lipschitz continuous away from the jump cube boundaries, but not in $\mathcal{W}^\infty(\Omega)$. This can be resolved by smoothing w^h. Indeed, we only have to replace the shape function Δ by its mollification. Again, this change will not affect the jump cubes and hence estimates of the energy of the jump set. Moreover, by choosing the mollification parameter small enough for each h, the convergences (33) can be maintained, and the energy bound (32) be replaced with

$$\int_\Omega W(\mathcal{E}\bar{v}^h(x))\,dx + \mathcal{H}^{m-1}(\operatorname{cl} J_{\bar{v}^h}) \leq \int_{\Omega'} W(\mathcal{E}u'(x))\,dx + c_m \mathcal{H}^{m-1}(J_{u'}) + \epsilon. \tag{34}$$

Finally, letting $\epsilon \searrow 0$, the existence of $\{u^i\}_{i=0}^\infty$ satisfying (28)–(29) follows from combining the estimates (33)–(34) between u' and $\bar{v}^h$, and the estimates (23)–(26) between u and u'. This concludes the proof. $\square$

Remark 1. If $u \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega; R^m)$ with $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{m\times m})} + \mathcal{H}^{m-1}(J_u) < \infty$, then the claim of Theorem 3 follows from the stronger approximation results of Cortesani and Toader [11], that show the existence of a sequence $\{u^i\}_{i=0}^\infty$ with J_{u^i} concentrated on finitely many $(m-1)$-dimensional simplices, and satisfying $u^i \to u$ strongly in $L^1(\Omega; \mathbb{R}^m)$ and $\nabla u^i \to \nabla u$ strongly in $L^p(\Omega; \mathbb{R}^{m\times m})$, along with

$$\limsup_{i\to\infty} \int_{A\cap J_{u^i}} \varphi(x, (u^i)^+, (u^i)^-, \nu_{J_{u^i}})\,d\mathcal{H}^{m-1} \leq \int_{A\cap J_u} \varphi(x, u^+, u^-, \nu_{J_u})\,d\mathcal{H}^{m-1}$$

for every $A \Subset \Omega$ and upper semicontinuous function $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times S^{m-1} \to [0, \infty)$ with $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$.

In fact, as Negri has observed in, e.g., [12, Proposition 2.4], this result of [11] may be partially extended to the BD case by combining with the approximation theorem of Chambolle [9, 10], which we have refined above. The claim is only for surface energies of the form $\varphi(x, a, b, \nu) = \varphi(\nu)$, and no proof is provided. It however does not appear to be based on providing an SBD counterpart to [11, Lemma 4.1], as this would show the case of general φ. Instead, in the isotropic case $\varphi(\nu) = \|\nu\|_2$, the claim seems to follow by directly employing (in [11, equation (5.2)]) the convergence of $\int_{J_{u^i}} \|\nu_{J_{u^i}}(x)\| \, d\mathcal{H}^{m-1} = \mathcal{H}^{m-1}(J_{u^i})$ that follows from (18). That this holds in the anisotropic case as well follows from Reshetnyak's continuity theorem.

Now, how this discussion relates to our work here is that Theorem 3 provides the missing full SBD counterpart to [11, Lemma 4.1], allowing full extension of [11, Theorem 3.1] to SBD in the case $P(u) < \infty$ (which is equivalent to $\|\mathcal{E}u\|_{L^2(\Omega;\mathbb{R}^{m\times m})} + \mathcal{H}^{m-1}(J_u) < \infty$, hence comparable to the assumption in the SBV case above).

3. The transport equation

3.1. The generalised formulation

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Let the final time $T > 0$ be specified, and set $\Omega^T := (0, T) \times \Omega$. Also fix constants $M_I, M_u, M_\tau \in (0, \infty)$. We then consider functions I and u in the spaces

$$
\begin{aligned}
&X_I := L^\infty_{M_I}(\Omega^T), \quad \text{and}\\
&X_u := \{u \in \mathrm{SBD}(\mathbb{R}^{n+1}) \mid u|\Omega^T = (1, b),\ u|(\mathbb{R}^{n+1} \setminus \operatorname{cl}\Omega^T) = 0,\ \|u\|_{L^\infty} \leq M_u\},
\end{aligned}
$$

implicitly extending I outside Ω^T by zero. We take $u \in \mathrm{SBD}(\mathbb{R}^{n+1})$ instead of $\mathrm{SBD}(\Omega^T)$, specifically restricting support, for notational purposes: we want J_u and $\mathrm{Div}^j\, u$ to include the jump over $\partial\Omega^T$, and to record initial conditions at time zero with $\mathrm{Div}^j\, u$.

We then define the $\mathcal{D}'(\mathbb{R}^{n+1})$ -valued functional corresponding to our extension $F(I, u; \tau) = 0$ of the transport equation for $u \in X_u$, $I \in X_I$, and $\tau \in L^1(\mathrm{Div}^j\, u)$ by

$$
\begin{aligned}
F(I, u; \tau)(\varphi) &:= \big(\mathrm{Div}(Iu) - I \operatorname{div} u\mathcal{L}^{n+1} - \tau\, \mathrm{Div}^j\, u\big)(\varphi)\\
&= -\int \langle \nabla\varphi, Iu\rangle \, d\mathcal{L}^{n+1} - \int \varphi I \operatorname{div} u \, d\mathcal{L}^{n+1} - \int \varphi\tau \, d\,\mathrm{Div}^j\, u, \quad (\varphi \in C^\infty_c(\mathbb{R}^{n+1})).
\end{aligned}
$$

Additionally, for use as a constraint in our image interpolation application of interest, we define the set-valued function $F : X_I \times X_u \rightrightarrows \mathcal{D}'(\mathbb{R}^{n+1})$ by

$$
F(I, u) := \{F(I, u; \tau) \mid \tau \in L^\infty_{M_\tau}(J_u)\}.
$$

The following example, already discussed in the Introduction, demonstrates the role of τ.

Example 1. Consider a moving ball (or other object) in one spatial dimension, as depicted in Figure 1. The domain $\Omega^T = (0, 1)^2$ can be divided into three subdomains: A_1, A_2, and B. In both A_1 and A_2, we have $u = (1, 0)$, while in B, we have $u = (1, v)$ for the speed v of movement of the ball. Recalling that J_u includes jumps on $\partial\Omega^T$, we have

$$
J_u = (\{0, 1\} \times [0, 1]) \cup (\partial A_1 \cap \partial B) \cup (\partial A_2 \cap \partial B) \cup (\partial B \cap \partial\Omega^T).
$$

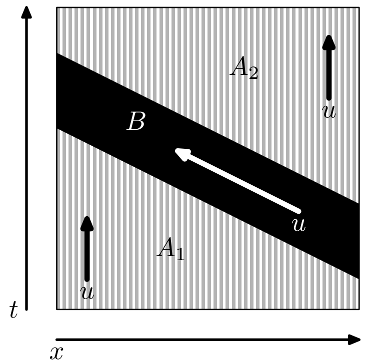

Figure 1: The situation of Example 1. The ball drawn in black reveals and hides background as it travels.

We easily observe that $\operatorname{div} u = 0$, while, denoting by ν the unit vector orthogonal orthogonal to $(1, v)$ satisfying $c := \langle (1,0), \nu \rangle > 0$, the jump part of the divergence is

$$\begin{aligned} \operatorname{Div}^j u =& \langle u^+ - u^-, \nu_{J_u} \rangle \mathcal{H}^1 \llcorner J_u \\ =& \mathcal{H}^1 \llcorner (\{0\} \times [0,1]) - \mathcal{H}^1 \llcorner (\{1\} \times [0,1]) \\ &+ c\big(\mathcal{H}^1 \llcorner (\partial A_2 \cap \partial B) - \mathcal{H}^1 \llcorner (\partial A_1 \cap \partial B)\big) \\ &- v\big(\mathcal{H}^1 \llcorner (\partial B \cap \{1\} \times [0,1]) - \mathcal{H}^1 \llcorner (\partial B \cap \{0\} \times [0,1])\big). \end{aligned} \tag{35}$$

The background intensity is constant in time, so in $A_1 \cup A_2$, the image $I(t,x) = \tau_0(x)$ for any given "initial condition" τ_0. In B, we have $I = \beta$, where we have taken the moving ball to have constant intensity β. Thus, $\operatorname{Div}(Iu) - I \operatorname{div} u = \operatorname{Div}(Iu) = \operatorname{Div}^j(Iu)$. Clearly then $\tau = d\operatorname{Div}^j(Iu)/d\operatorname{Div}^j u$ satisfies $F(I, u; \tau) = 0$, provided $\operatorname{Div}^j(Iu) \ll \operatorname{Div}^j u$. Let us calculate $\operatorname{Div}^j(Iu)$ explicitly. Abusing notation by writing $\tau_0(t,x) = \tau_0(x)$, we have

$$\begin{aligned} \operatorname{Div}^j(Iu) =& \langle I^+ u^+ - I^- u^-, \nu_{J_u} \rangle \mathcal{H}^1 \llcorner J_u \\ =& \tau_0\big(\mathcal{H}^1 \llcorner (\{0\} \times [0,1]) - \mathcal{H}^1 \llcorner (\{1\} \times [0,1])\big) \\ &+ \tau_0 c\big(\mathcal{H}^1 \llcorner (\partial A_2 \cap \partial B) - \mathcal{H}^1 \llcorner (\partial A_1 \cap \partial B)\big) \\ &- \beta v\big(\mathcal{H}^1 \llcorner (\partial B \cap \{1\} \times [0,1]) - \mathcal{H}^1 \llcorner (\partial B \cap \{0\} \times [0,1])\big). \end{aligned} \tag{36}$$

Comparing (35) to (36), we find, as expected, that $\tau(t,x) = \tau_0(x)$ on $\{0,1\} \times [0,1]$, and $\tau(t,x) = \beta$ on $\partial B \cap \partial \Omega^T$. On $(\partial A_1 \cup \partial A_2) \cap B$, we also have $\tau(t,x) = \tau_0(x)$. In this particular example, with the normal ν always orthogonal to u on one side of jump set, τ thus completely describes "what goes of I into a sink, or comes from a source". Furthermore, τ is clearly bounded when the background intensity is.

In the following, we study various properties of the function F. We begin with showing that F is continuous in the set-valued sense of being both inner- and outer-semicontinuous. After that we study existence and subsequently uniqueness of solutions to the inclusion $0 \in F(I, u)$.

3.2. Continuity

We now prove the following theorem that establishes the inner- and out outer-semicontinuities of F.

Theorem 4. *Suppose $\{I^i\}_{i=0}^\infty \subset X_I$ converges to $I \in X_I$ pointwise a.e. in Ω^T, and $\{u^i\}_{i=0}^\infty \subset X_u$ converges to $u \in X_u$ in the sense*

$$u^i \to u \text{ strongly in } L^1(\Omega^T), \tag{37}$$

$$\operatorname{div} u^i \rightharpoonup \operatorname{div} u \text{ weakly in } L^1(\Omega^T), \tag{38}$$

$$\operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \operatorname{Div}^j u \text{ weakly* in } \mathcal{M}(\mathbb{R}^{n+1}), \text{ and} \tag{39}$$

$$\lim_{i\to\infty} |\operatorname{Div}^j u^i|(\mathbb{R}^{n+1}) = |\operatorname{Div}^j u|(\mathbb{R}^{n+1}). \tag{40}$$

Then

$$\limsup_{i\to\infty} F(I^i, u^i) \subset F(I, u) \qquad \text{weakly* in } \mathcal{D}'(\mathbb{R}^{n+1}). \tag{41}$$

Suppose, moreover, that $\mathcal{H}^n(J_u) < \infty$. Then for every $\tau \in L^\infty_{M_\tau}(J_u)$, there exist $\tau^i \in L^\infty_{M_\tau}(J_{u^i})$, ($i = 0, 1, 2, \ldots$), such that

$$F(I^i, u^i; \tau^i) \overset{*}{\rightharpoonup} F(I, u; \tau) \qquad \text{weakly* in } \mathcal{D}'(\mathbb{R}^{n+1}). \tag{42}$$

In particular

$$\liminf_{i\to\infty} F(I^i, u^i) \supset F(I, u) \qquad \text{weakly* in } \mathcal{D}'(\mathbb{R}^{n+1}).$$

Observe that (37)–(39) follow from Theorem 2. We will return to conditions ensuring (40) in Section 4. The most important consequence of the theorem for our purposes is the following.

Corollary 1. *Suppose $\{I^i\}_{i=0}^\infty \subset X_I \cap \mathrm{BV}(\mathbb{R}^{n+1})$ converges to $I \in X_I \cap \mathrm{BV}(\mathbb{R}^{n+1})$ weakly in $\mathrm{BV}(\mathbb{R}^{n+1})$, and $\{u^i\}_{i=0}^\infty \subset X_u$ converges to $u \in X_u$ in the sense (37)–(40). Then $0 \in F(I^i, u^i)$ for $i = 0, 1, 2, \ldots$ implies $0 \in F(I, u)$.*

Remark 2. Under the above assumption that $I \in X_I \cap \mathrm{BV}(\mathbb{R}^{n+1})$, Proposition 5 in the Appendix implies that the values of F are, in fact, measures, not just distributions.

Proof of Theorem 4. The outer-semicontinuity (41) is established by showing continuity for each of the terms $\operatorname{Div}(Iu)$, $I \operatorname{div} u\mathcal{L}^{n+1}$, and $\tau \operatorname{Div}^j u$ separately. We first tackle $\operatorname{Div}(Iu)$. By assumption, we have $u^i \to u$ strongly in $L^1(\Omega^T; \mathbb{R}^{n+1})$, and $I^i \to I$ pointwise a.e. in Ω^T. As $\mathcal{L}^n(\Omega^T) < \infty$ and $\|I^i\|_{L^\infty} \leq M_I$, we thus have $I^i u^i \rightharpoonup Iu$ weakly in $L^1(\Omega^T)$; see, e.g., [21, Proposition 2.61]. Since Div is a continuous linear operator between the weak topology on L^1 and the weak* topology of distributions, it follows that $\operatorname{Div}(I^i u^i) \overset{*}{\rightharpoonup} \operatorname{Div}(Iu)$ weakly* as distributions,

Next we consider $I \operatorname{div} u\mathcal{L}^{n+1}$. By (38), we have $\operatorname{div} u^i \rightharpoonup \operatorname{div} u$ weakly in $L^1(\Omega^T)$. From the pointwise a.e. convergence of I^i we therefore get again that $I^i \operatorname{div} u^i \rightharpoonup I \operatorname{div} u$ weakly in $L^1(\Omega^T)$. In particular, $I^i \operatorname{div} u^i \mathcal{L}^n \overset{*}{\rightharpoonup} I \operatorname{div} u\mathcal{L}^n$ weakly* as measures, hence as distributions.

Finally, we have to study subsequences of $\{\tau^i \operatorname{Div}^j u^i\}_{i=0}^\infty$ convergent weakly* as distributions. From (40), $u \in \mathrm{BD}(\Omega^T)$, and $\tau^i \in L^\infty_{M_\tau}(J_{u^i})$ we however observe that such sequences are bounded, hence measures (see, e.g., [22]), and may be assumed to converge weakly* as measures. What we therefore have to establish is that given a subsequence of $\{\tau^i \operatorname{Div}^j u^i\}_{i=0}^\infty$, unrelabelled, such that $\tau^i \operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \nu$ weakly* in $\mathcal{M}(\mathbb{R}^{n+1})$, then $\nu = \tau \operatorname{Div}^j u$ for some $\tau \in L^\infty_{M_\tau}(J_u)$. But, for any $\varphi \in C_c(\mathbb{R}^{n+1})$, we may calculate

$$\begin{aligned}
\nu(\varphi) &= \lim_{i\to\infty} \tau^i \operatorname{Div}^j u^i(\varphi) \leq \limsup_{i\to\infty} |\tau^i| |\operatorname{Div}^j u^i|(|\varphi|) \\
&\leq M_\tau \limsup_{i\to\infty} |\operatorname{Div}^j u^i|(|\varphi|) = M_\tau |\operatorname{Div}^j u|(|\varphi|),
\end{aligned}$$

where the last step follows from (40). This shows that $|\nu| \le M_\tau |\operatorname{Div}^j u|$, allowing us to conclude the proof of (41).

Now we have to show (42). We have already shown the continuity of $\operatorname{Div}(Iu)$ (as a distribution) and of $I \operatorname{div} u\mathcal{L}^n$. Therefore, to complete the proof, it suffices to show the existence of some $\tau^i \in X_\tau$ such that $\tau_i \operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \tau \operatorname{Div}^j u$ weakly* in $\mathcal{M}(\mathbb{R}^{n+1})$.

Towards this end, we let $\rho(x) := \chi_{B(0,1)}(x) \exp(-1/(1-\|x\|^2))$ be the standard mollifier on $\mathbb{R}^{n+1}$, and set $\rho_\epsilon(x) := \epsilon^{-n}\rho(x/\epsilon)$. Mind the factor ϵ^{-n} instead of $\epsilon^{-(n+1)}$. Then we set

$$\tau_\epsilon(x) := C^{-1}[\rho_\epsilon * (\tau\mathcal{H}^n \llcorner J_u)](x) = C^{-1} \int_{J_u} \rho_\epsilon(x-y)\tau(y)\, d\mathcal{H}^n(y)$$

for a yet undetermined constant C. We then observe that by choosing the constant C appropriately, when the n-dimensional density exists, we have

$$\lim_{\epsilon \searrow 0} \tau_\epsilon(x) = \Theta_n(\tau\mathcal{H}^n \llcorner J_u, x) \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathbb{R}^{n+1}. \tag{43}$$

Indeed, let us write $\rho(x) = \int_0^{\exp(-1)} \chi_{\rho \le t}(x)\, dt$. Minding that $\{x \in \mathbb{R}^{n+1} \mid \rho(x) \le t\} = B(0, f(t))$ for some decreasing $f : [0,e] \to [0,1]$, we get

$$\begin{aligned} C\tau_\epsilon(x) &= \epsilon^{-n} \int_{J_u} \int_0^{\exp(-1)} \chi_{B(0,f(r))}((x-y)/\epsilon)\, dr \tau(y)\, d\mathcal{H}^n(y) \\ &= \int_0^{\exp(-1)} [f(r)]^n \Big([\epsilon f(r)]^{-n} (\tau\mathcal{H}^n \llcorner J_u)(B(x, \epsilon f(r))) \Big)\, dr. \end{aligned}$$

By application of Fatou's lemma, and the fact that $\Theta_n(\tau\mathcal{H}^n \llcorner J_u, x)$ exists for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^{n+1}$ by rectifiability, we deduce (43) with $C := \int_0^{\exp(-1)} [f(r)]^n\, dr$.

But, now, employing our assumption $\mathcal{H}^n(J_u) < \infty$, the jump set J_u is $\mathcal{H}^n$-rectifiable. Therefore, $\Theta_n(\tau\mathcal{H}^n \llcorner J_u, x) = \tau(x)$ for $\mathcal{H}^n$-a.e. $x \in J_u$, and $\Theta_n(\tau\mathcal{H}^n \llcorner J_u, x) = 0$ for $\mathcal{H}^n$-a.e. $x \notin J_u$; see, e.g., [23, 2]. So, in summary, we get from (43) that $\tau_\epsilon(x) \to \tau(x)$ as $\epsilon \searrow 0$ for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^{n+1}$.

Let us then set $\bar\tau_\epsilon(x) = \max\{\min\{\tau_\epsilon(x), M_\tau\}, -M_\tau\}$. Still we have $\bar\tau_\epsilon \in C_c(\mathbb{R}^{n+1})$, and, minding that $\tau \in L^\infty_{M_\tau}(J_u)$, also $\bar\tau_\epsilon(x) \to \tau(x)$ as $\epsilon \searrow 0$ for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^{n+1}$. In consequence,

$$\bar\tau_\epsilon \operatorname{Div}^j u \overset{*}{\rightharpoonup} \tau \operatorname{Div}^j u \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1}), \quad (\epsilon \searrow 0),$$

Moreover, by the weak* convergence of $\operatorname{Div}^j u^i$ to $\operatorname{Div}^j u$, for any $\epsilon > 0$, we have

$$\bar\tau_\epsilon \operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \bar\tau_\epsilon \operatorname{Div}^j u \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1}), \quad (i \to \infty).$$

Next we observe that all the involved measures lie in a bounded subset of $\mathcal{M}(\mathbb{R}^{n+1})$. Hence the weak* convergences are given by a metric. We may thus perform a diagonal construction yielding $\epsilon_i > 0$ and $\tau_i := \bar\tau_{\epsilon_i}|J_{u^i} \in L^\infty_{M_\tau}(J_{u^i})$, $(i = 0, 1, 2, \ldots)$, such that $\tau_i \operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \tau \operatorname{Div}^j u$. □

The next proposition shows that provided $\{I^i\}_{i=0}^\infty$ converge weakly* in $L^\infty(\Omega^T)$ (as is the case for a subsequence when $I \in X_I$), then the functional F is closed also with respect to specific restricted mollifications u^{ϵ_i} of u with weaker convergence properties than (37)–(40).

Proposition 1. *Suppose $u \in X_u$, and let $\{\eta_\epsilon\}_{\epsilon>0}$ be a family of mollifiers on $\mathbb{R}^{n+1}$. Let $Q := (0,T) \times \mathbb{R}^n$, and define $u^\epsilon : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by*

$$u^\epsilon := \chi_{\operatorname{cl} Q} \cdot (\eta_\epsilon * \bar{u}), \qquad \text{where} \qquad \bar{u} := \begin{cases} u, & x \in \Omega^T, \\ (1,0), & x \in \mathbb{R}^{n+1} \setminus \Omega^T. \end{cases} \tag{44}$$

Suppose that $\{I^i\}_{i=0}^\infty \subset L^\infty_{M_I}(Q)$ converges weakly in $L^\infty(Q)$ to $I \in L^\infty_{M_I}(Q)$, and that $M_I \leq M_\tau$. Then, letting $F_\delta(I,u) := \{F(I,u;\tau) \mid \tau \in L^\infty_{M_\tau}(\{0,T\} \times (\Omega + B(0,\delta)))\}$ and taking a sequence $\epsilon^i \searrow 0$, we have*

$$\limsup_{i\to\infty} F_{2\epsilon_i}(I^i, u^{\epsilon_i}) \subset F(I,u) \qquad \text{weakly* in } \mathcal{D}'(\mathbb{R}^{n+1}).$$

In particular, suppose I^i are solutions of the classical transport equation for velocity field u^{ϵ_i}, and initial condition $\tau_0^i \in L^\infty_{M_\tau}(\{0\} \times \mathbb{R}^n)$. Suppose, moreover, that τ_0^i have support on $\{0\} \times (\Omega + B(0,2\epsilon_i))$, and are convergent weakly in $L^\infty(\{0\} \times \Omega)$ to τ_0. Then $F(I,u;\tau) = 0$ for some $\tau \in L^\infty_{M_\tau}(J_u)$ satisfying $\tau = \tau_0$ on $\{0\} \times \Omega$.*

(The restriction of τ in the definition of F_δ is only needed because $\operatorname{Div}^j u^\epsilon$ has unbounded support ∂Q. We could alternatively restrict u^ϵ to $[0,T] \times (\Omega + B(0,2\epsilon_i))$, assuming that $\operatorname{supp} \eta_\epsilon \Subset B(0,2\epsilon)$.)

Proof. First of all, we claim that

$$\operatorname{div} u^\epsilon = \eta_\epsilon * (\operatorname{Div} u \llcorner Q) \quad \text{on } Q. \tag{45}$$

To see this, we observe that on $\{0,T\} \times \Omega$, with normal $\nu = (1,0)$, we have $\langle \bar{u}^\pm, \nu \rangle = 1$. Therefore, we necessarily have $\operatorname{Div}^j \bar{u} \llcorner (\{0,T\} \times \Omega) = 0$. Then we note that on $(0,T) \times \partial\Omega$ with normal ν satisfying $\langle \nu, (1,0) \rangle = 0$ and pointing out of Ω^T, we get $\langle \bar{u}^+, \nu \rangle = 0$ and $\langle \bar{u}^-, \nu \rangle = \langle u^-, \nu \rangle$. Thus the jump divergence is unaffected: $\operatorname{Div}^j \bar{u} \llcorner ((0,T) \times \partial\Omega) = \operatorname{Div}^j u \llcorner ((0,T) \times \partial\Omega)$. As $\bar{u} = 0$ is constant outside Ω^T, it follows that $\operatorname{Div} \bar{u} = \operatorname{Div} u \llcorner Q$. This shows (45).

Since $\operatorname{Div} u \llcorner Q = \operatorname{div} u \mathcal{L}^{n+1} + \operatorname{Div}^j u \llcorner Q$, we now have for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ that

$$\int_Q \varphi I^i \operatorname{div} u^{\epsilon_i} \, dx = \int_Q \varphi I^i (\eta_{\epsilon_i} * \operatorname{div} u) \, dx + \int_Q \varphi I^i (\eta_{\epsilon_i} * (\operatorname{Div}^j u \llcorner Q)) \, dx, \quad (i = 0,1,2,\ldots). \tag{46}$$

We next study the convergence properties of the two terms on the right hand side of (46). Because $|\eta_{\epsilon_i} * (\operatorname{Div}^j u \llcorner Q) \mathcal{L}^{n+1}|(Q) \leq |\operatorname{Div}^j u|(\mathbb{R}^{n+1}) < \infty$ by standard properties of mollification, and $\|I^i\|_{L^\infty} \leq M_I < \infty$, it follows that there is a subsequence of $\{(I^i, \epsilon_i)\}_{i=0}^\infty$, unrelabelled, such that

$$I^i (\eta_{\epsilon_i} * (\operatorname{Div}^j u \llcorner Q)) \overset{*}{\rightharpoonup} \nu \qquad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1})$$

for some finite Radon measure ν concentrated on $\operatorname{cl} Q$. Now we observe that by standard mollification results

$$|\eta_{\epsilon_i} * (\operatorname{Div}^j u \llcorner Q) \mathcal{L}^{n+1}| \overset{*}{\rightharpoonup} |\operatorname{Div}^j u \llcorner Q| \qquad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1}).$$

As in the the proof of Theorem 4, it therefore follows that there exists $\tau_{\text{int}} \in L^\infty_{M_I}(\operatorname{Div}^j u \llcorner Q)$ such that

$$\nu = \tau_{\text{int}} \operatorname{Div}^j u \llcorner Q.$$

Next, we note that as $\eta_{\epsilon_i} * \operatorname{div} u \to \operatorname{div} u$ strongly in $L^1(Q)$, and $I^i \overset{*}{\rightharpoonup} I$ weakly* in $L^\infty(Q)$, we have $I^i(\eta_{\epsilon^i} * \operatorname{div} u) \rightharpoonup I \operatorname{div} u$ weakly in $L^1(Q)$. The decomposition (46) thus yields

$$\int_Q \varphi I^i \operatorname{div} u^{\epsilon_i}\, dx \to \int_Q \varphi I \operatorname{div} u\, dx + \int \varphi \tau_{\text{int}}\, d\operatorname{Div}^j u \llcorner Q, \quad (\varphi \in C_c^\infty(\mathbb{R}^{n+1})). \tag{47}$$

Similarly, as $u^{\epsilon_i} \to u$ strongly in $L^1(Q)$, we also have

$$\int_Q \langle \nabla\varphi, I^i u^{\epsilon_i}\rangle\, dx \to \int_Q \langle \nabla\varphi, I u\rangle\, dx, \quad (\varphi \in C_c^\infty(\mathbb{R}^{n+1})). \tag{48}$$

Next, we note that $\operatorname{Div}^j u^\epsilon = \mathcal{H}^n \llcorner \{0\} \times \mathbb{R}^n - \mathcal{H}^n \llcorner \{T\} \times \mathbb{R}^n$ and $\operatorname{Div}^j u \llcorner \partial Q = \mathcal{H}^n \llcorner \{0\} \times \Omega - \mathcal{H}^n \llcorner \{T\} \times \Omega$. Hence, given $\tau^i \in L^\infty_{M_\tau}(\{0,T\} \times (\Omega + B(0, 2\epsilon_i)))$, $(i = 0,1,2,\ldots)$, we establish the existence of some $\tau_\partial \in L^\infty_{M_\tau}(\{0,T\} \times \Omega)$, such that for an unrelabelled subsequence

$$\tau^i \operatorname{Div}^j u^{\epsilon_i} \overset{*}{\rightharpoonup} \tau_\partial \operatorname{Div}^j u \llcorner \partial Q \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1}). \tag{49}$$

Combining (47)–(49), we now obtain

$$\begin{aligned}\int_Q \langle \nabla\varphi, I^i u^{\epsilon_i}\rangle\, dx + \int_Q \varphi I^i \operatorname{div} u^{\epsilon_i}\, dx + \int_{\operatorname{cl} Q} \varphi \tau^i\, d\operatorname{Div}^j u^{\epsilon_i} \\ \to \int_Q \langle \nabla\varphi, I u\rangle\, dx + \int_Q \varphi I \operatorname{div} u\, dx + \int_{\operatorname{cl} Q} \varphi\tau\, d\operatorname{Div}^j u, \quad (\varphi \in C_c^\infty(\mathbb{R}^{n+1})),\end{aligned} \tag{50}$$

for $\tau := \tau_{\text{int}} + \tau_\partial$. Because τ_∂ is supported on ∂Q and τ_{int} on $J_u \cap Q$ with $\partial Q \cap Q = \emptyset$, we have $\|\tau\|_{L^\infty(\operatorname{Div}^j u)} \le M_\tau$. We may therefore conclude from (50) that any weak* limit $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ of a subsequence of $\{v^i\}_{i=0}^\infty$ with $v_i \in F_{2\epsilon_i}(I^i, u^{\epsilon_i})$, satisfies $v \in F(I,u)$. This concludes the proof of the first part of the proposition.

It remains to study the case with $\{I^i\}_{i=0}^\infty$ solutions to the classical transport equation with initial condition τ_0^i and velocity field u^{ϵ_i}. Regarding this, we observe from (49) that $\tau = \tau_\partial = \tau_0$ on $\{0\} \times \Omega$ since $\tau^i|(\{0\} \times \Omega) = \tau_0^i \overset{*}{\rightharpoonup} \tau_0$ weakly* in $L^\infty(\{0\} \times \Omega)$. This completes the proof. □

Remark 3. The above outer-semicontinuity results prove some degree of stability of the inclusion $0 \in F(I,u)$, however largely ignoring any "initial conditions on discontinuities in space-time" for I. As this topic merits some more discussion, we will return to it in Remark 5 following our existence theorem.

3.3. *A technical lemma*

We will need the following lemma for the existence and uniqueness results to follow. One of its consequences is that even without assuming I to be of bounded variation, solutions I of $0 \in F(I,u)$ (when $\mathcal{H}^n(J_u) < \infty$) have one-sided Lebesgue limits on J_u when u is not parallel to J_u. To state the lemma, and for later use, we define

$$P_u^\pm := \{x \in J_u \mid \langle u^\pm(x), \pm\nu_{J_u}(x)\rangle > 0\}, \quad \text{and} \tag{51}$$
$$N_u^\pm := \{x \in J_u \mid \langle u^\pm(x), \pm\nu_{J_u}(x)\rangle < 0\}. \tag{52}$$

Lemma 3. *Suppose $F(I, u; \tau) = 0$ for some $I \in X_I$, $u \in X_u$, and $\tau \in L^1(\operatorname{Div}^j u)$. Then the one-sided Lebesgue limits $I^\pm$ exist $\mathcal{H}^n$-a.e. on $P_u^\pm \cup N_u^\pm$, and $(Iu)^\pm = I^\pm u^\pm$ (a.e.). Moreover, defining $I^\pm$ arbitrarily on $J_u \setminus (N_u^\pm \cup P_u^\pm)$, we have $\operatorname{Tr}^\pm(Iu, J_u) = I^\pm \langle u^\pm, \nu_{J_u} \rangle$, and*

$$\tau \langle u^+ - u^-, \nu_{J_u} \rangle = \langle I^+ u^+ - I^- u^-, \nu_{J_u} \rangle \quad \mathcal{H}^n\text{-a.e. on } J_u. \tag{53}$$

Proof. First of all, since u is of bounded deformation, we observe that $\operatorname{Tr}^\pm(u, J_u) = \langle u^\pm, \nu_{J_u} \rangle$ on J_u ($\mathcal{H}^n$-a.e.); see [1]. Next we note that the measures $I \operatorname{div} u \mathcal{L}^{n+1}$ and $\tau \operatorname{Div}^j u$ are bounded under present assumptions. Hence it follows from $F(I, u; \tau) = 0$ that $\operatorname{Div}(Iu)$ is a bounded measure. We may therefore apply [13, Theorem 6.2] to show that $I^\pm$ exists $\mathcal{H}^n$-a.e. on J_u when $\operatorname{Tr}^\pm(u, J_u) \neq 0$, i.e., on $P_u^\pm \cup N_u^\pm$. In fact

$$I^\pm = \operatorname{Tr}^\pm(Iu, J_u) / \operatorname{Tr}^\pm(u, J_u) \quad \mathcal{H}^n\text{-a.e. on } P_u^\pm \cup N_u^\pm. \tag{54}$$

Since u and I are bounded, and $u^\pm$ and $I^\pm$ exist, it now follows easily from the definition (6) of the one-sided Lebesgue limit that $(Iu)^\pm = I^\pm u^\pm$ on $P_u^\pm \cup N_u^\pm$ ($\mathcal{H}^n$-a.e.).

It remains to show (53). It follows from (54) that

$$\operatorname{Tr}^\pm(Iu, J_u) = I^\pm \langle u^\pm, \nu_{J_u} \rangle \quad \mathcal{H}^n\text{-a.e. on } P_u^\pm \cup N_u^\pm. \tag{55}$$

Next we deduce from, e.g., [13, Theorem 4.2] (see (116)), that $\operatorname{Tr}^\pm(Iu, J_u) = 0$ when $\operatorname{Tr}^\pm(u, J_u) = 0$. That is to say

$$\operatorname{Tr}^\pm(Iu, J_u) = 0 \quad \mathcal{H}^n\text{-a.e. on } J_u \setminus (N_u^\pm \cup P_u^\pm). \tag{56}$$

Finally, minding that $F(I, u; \tau) = 0$, we have $\operatorname{Div}(Iu) \llcorner J_u = \tau \operatorname{Div}^j u$. Therefore, e.g., [13, Proposition 3.4] shows that

$$\tau \operatorname{Div}^j u = \operatorname{Div}(Iu) \llcorner J_u = (\operatorname{Tr}^+(Iu, J_u) - \operatorname{Tr}^-(Iu, J_u)) \mathcal{H}^n \llcorner J_u. \tag{57}$$

Defining $I^\pm$ arbitrarily on $J_u \setminus (N_u^\pm \cup P_u^\pm)$, we now deduce (53) from (55)–(57). □

3.4. Existence

We now provide a weak existence result, based on the approximation of Theorem 3. Of course, any constant function I is always a solution to $0 \in F(I, u)$ given $u \in X_u$ and no boundary conditions. In Theorem 5 below, we show that we can at least in a very weak distributional sense, control the traces of $I \in X_I$ on the one-sided "source parts" (see Figure 2)

$$L_u^\pm := P_u^\pm \cap J_u^{\text{div}} \tag{58}$$

of the jump set, where $P_u^\pm$ is defined in (51), and

$$J_u^{\text{div}} := \{x \in J_u \mid \langle u^+(x) - u^-(x), \nu_{J_u}(x) \rangle \neq 0\}.$$

For simplicity, here and throughout this section, *we assume without loss of generality* that ν_{J_u} is chosen continuously along each of the at most countably many C^1 surfaces $\{\Gamma_i\}_{i=1}^\infty$ containing J_u (The choice is to be approximately continuous on each surface Γ_i over the $\mathcal{H}^n$-negligible set where these surfaces intersect.)

We begin with an existence result for more regular functions u. Although long and tedious to prove, the proposition is rather obvious and most of the arguments quite standard for the transport

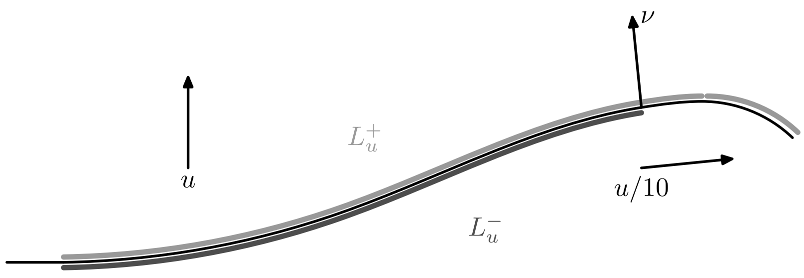

Figure 2: The "source parts" $L_u^\pm$ of J_u.

equation, although some changes in techniques are in order, because we need to piece together the solution from flows originating from multiple surfaces. We have, however, been unable to find an existing directly applicable result, so we provide an almost self-contained proof, skimming over some of the arguments that follow exactly as in the classical case. Most of the work in the long proof lies in showing that $(\mathrm{Div}(Iu) - I \operatorname{div} u) \llcorner (J \setminus J_u^{\mathrm{div}}) = 0$.

Proposition 2. *Suppose $u = (1, b) \in \mathcal{W}^\infty(\mathbb{R}^{n+1}) \cap X_u$. Let $Y^\pm \in L^\infty_{M_I}(L^\pm)$ for some $M_I \geq 0$ and Borel sets $L^\pm \subset L_u^\pm$. Then there exists a solution $I \in L^\infty_{M_I}(\Omega^T)$ and $\tau \in L^1(\mathrm{Div}^j(u))$ to $F(I, u; \tau) = 0$ with $I^\pm = Y^\pm$ on $L^\pm$ and $I^+ = I^-$ on $J_u^{\mathrm{div}} \setminus (L^+ \cup L^-)$.*

Proof. We divide the proof into four steps: (Step 1) Construction of flows $X^\pm$ and of I, (Step 2) showing that $0 \in F(I, u)$ holds along with (Step 3) $I^\pm = Y^\pm$ on $L^\pm$, subject to (Step 4) the properties (73), (76), and (75) of the auxiliary maps h and g. We begin, however, by establishing some more notation used throughout the proof. First of all, we denote by J the discontinuity set in the definition of $\mathcal{W}^\infty(\mathbb{R}^{n+1})$, with normal ν_J (chosen continuously, as in the discussion above). We have $J_u \subset J$, but this inclusion may be strict, even satisfying $\mathcal{H}^n(J \setminus J_u) > 0$. Nevertheless, by the definition of the jump set, we have

$$J_u^{\mathrm{div}} = \{x \in J \mid \langle u^+(x) - u^-(x), \nu_J(x) \rangle \neq 0\}.$$

We then denote by $J_0 \subset J$ the set of $x \in J$ where we have the existence of $\rho > 0$ such that the ball $B(x, \rho)$ is split into two open halves $U^\pm(x)$ by $\operatorname{cl} U \cap \Gamma$ for one of the C^1 surfaces Γ containing J, and such that $(\operatorname{cl} B(x, \rho) \setminus \Gamma) \cap J = \emptyset$. (The signs denoting sides are taken consistent with $u^\pm$ and ν_J.) Clearly J_0 is open relative to $\operatorname{cl} J$, and $\mathcal{H}^n(\operatorname{cl} J \setminus J_0) = 0$. Finally, we will be extensively working on the sets

$$\begin{aligned} P_0^\pm &:= \{x \in J_0 \mid \langle u^\pm(x), \pm\nu_J(x) \rangle > 0\}, \\ N_0^\pm &:= \{x \in J_0 \mid \langle u^\pm(x), \pm\nu_J(x) \rangle < 0\}, \quad \text{and} \\ Z_0^\pm &:= J_0 \setminus \operatorname{cl}(P_0^\pm \cup N_0^\pm). \end{aligned}$$

Step 1: Construction. By classical results, at any point $(t_0, x_0) \in \Omega^T \setminus \operatorname{cl} J$, there exists locally on an interval around t_0, a unique solution γ of

$$\gamma'(t) = b(t, \gamma(t)), \quad \gamma(t_0) = x_0. \tag{59}$$

Such a solution may further be uniquely extended to reach the set $\operatorname{cl} J$ at both ends; see, e.g., [6]. (Recall that J includes the initial and final boundaries $\{0, T\} \times \Omega$, as well as other parts of $\partial\Omega^T$

where u is not orthogonal to the normal of $\partial\Omega^T$.) At each $(t_0, x_0) \in J_0$, on the other hand, we may find unique solutions γ^+ and γ^- to (59) in $U^+(t_0, x_0)$ and $U^-(t_0, x_0)$ (choosing in (59) $(1, b) = u^\pm$ on J_0). Therefore, at any $(t_0, x_0) \in (\Omega^T \setminus \mathrm{cl}\, J) \cup (P_0^\pm \cup N_0^\pm)$, we may identify a unique curve $\gamma^\pm_{(t_0,x_0)} : [a, q] \to \mathbb{R}^n$ satisfying (59) on some interval $[a, q) := [a^\pm(t_0, x_0), q^\pm(t_0, x_0)) \ni t_0$, and, as we will show shortly, the inclusions

$$\gamma^\pm_{(t_0,x_0)}((a, q)) \subset \Omega^T \setminus \mathrm{cl}\, J \tag{60}$$

$$\gamma^\pm_{(t_0,x_0)}(a) \in \mathrm{cl}\, J \setminus ((N_0^+ \cup Z_0^+) \cap (N_0^- \cup Z_0^-)), \quad \text{and} \tag{61}$$

$$\gamma^\pm_{(t_0,x_0)}(q) \in \mathrm{cl}\, J \setminus ((P_0^+ \cup Z_0^+) \cap (P_0^- \cup Z_0^-)). \tag{62}$$

Moreover, $\gamma^+_{(t_0,x_0)} = \gamma^-_{(t_0,x_0)}$ if $(t_0, x_0) \notin \mathrm{cl}\, J$.

To establish (61), (62), we make the following observations: We let $(t, x) \in P_0^+$, and set $z := \nu_J(t, x)$. Observe that by taking $\delta > 0$ small enough, we may assume $\langle z, u(y)\rangle > 0$ for $y \in U^+(t, x)$. Suppose then that we have a solution γ of (59) in $U^+(t, x)$, defined on $(t_0, t]$, and satisfying $\gamma(t) = x$. We then have

$$\langle z, (t, x) - (s, \gamma(s))\rangle = \langle z, \int_s^t u(r, \gamma(r))\, dr\rangle > 0 \quad \text{for } s \in (t_0, t).$$

In particular, $\lim_{s\nearrow t}\langle z, (t, x) - (s, \gamma(s))\rangle/(t - s) = \langle z, u^+(t, x)\rangle > 0$. On the other hand, minding that $-z$ is the normal to the tangent cone of $U^+(t, x)$ at (t, x), we have $\lim_{s\nearrow t}\langle z, (t, x) - (s, \gamma(s))\rangle/(t - s) \leq 0$. This contradiction shows that no solution can reach $(t, x) \in P_0^+$ from $U^+(t, x)$. Next, we note that any solution to (59) in $\mathrm{cl}\, U^+(t_0, x_0)$ with $(t_0, x_0) \in Z_0^+$ will locally stay on the manifold Z_0^+. This is because the field u^+ on Z_0^+ is locally orthogonal to the normal, so there is a solution curve γ on the manifold, and solutions on $\mathrm{cl}\, U^+$ are unique. Again this shows that no solution can reach $(t, x) \in Z_0^+$ from $U^+(t, x)$. As similar results hold on $U^-(t, x)$ for $(t, x) \in P_0^-$ and $(t, x) \in Z_0^-$, we conclude with (62). Similarly, working with $N_0^\pm$ and traversing γ "in reverse" we establish (61).

Let us now set

$$G^\pm := \bigcup\{(t, q^\pm(t, x)) \times \{(t, x)\} \mid (t, x) \in P_0^\pm\}.$$

Then, based on what we have shown so far, we may define on G^+ and G^- the respective flows X^+ and X^-, satisfying at $(t, (t_0, x_0)) \in G^\pm$ the conditions

$$\begin{gathered} \partial_t X^\pm(t, (t_0, x_0)) = b^\pm(t, X^\pm(t, (t_0, x_0))), \\ X^\pm(t_0, (t_0, x_0)) = x_0, \quad \text{and} \\ X^\pm(q^\pm(t, x), (t_0, x_0)) \in \mathrm{cl}\, J \setminus ((P_0^+ \cup Z_0^+) \cap (P_0^- \cup Z_0^-)). \end{gathered} \tag{63}$$

If we now set

$$E^\pm := \{(t, X(t, (t_0, x_0))) \mid (t, (t_0, x_0)) \in G^\pm\},$$

then $E^+ \cap E^- = \emptyset$, and by (61), $\Omega \setminus (E^+ \cup E^- \cup J_0)$ consists of points (t, x) with $\gamma^\pm_{(t,x)}(a(t, x))$ in the $\mathcal{H}^n$-negligible sets $\mathrm{cl}\, J \setminus (P_0^\pm \cup N_0^\pm \cup Z_0^\pm)$. Minding that we want to show the existence of I with traces $Y^\pm$ on $L^\pm$, we may therefore largely limit our attention to the sets E^+ an E^-.

Before defining I shortly, we introduce the auxiliary maps

$$\begin{aligned} h^\pm(t, x) &:= \big(a^\pm(t, x), \gamma^\pm_{(t,x)}(a^\pm(t, x), x)\big), \quad \text{and} \\ g^\pm(t, x) &:= \big(q^\pm(t, x), \gamma^\pm_{(t,x)}(q^\pm(t, x), x)\big). \end{aligned}$$

These give the initial and final points in space-time of the solution curve $\gamma^{\pm}_{(t,x)}$. Observe that $g^{\pm}(t,x) = \big(q^{\pm}(t,x), X^{\pm}(q^{\pm}(t,x),(t,x))\big)$ on $P^{\pm}_0$. Also $h^{\pm}(t,x) = [X^{\pm}(t,\cdot)]^{-1}(x)$ when $(t,x) \in E^{\pm}$, but this does not apply when $t = q^{\pm}(t,x)$. Moreover, on $\Omega^T \setminus \operatorname{cl} J$ we have $f^+ = f^-$ for $f = a, q, g, h$. We therefore often drop the sign superscript when it makes no difference.

Finally, we set

$$I(t,x) := \begin{cases} \widetilde{Y}^+(h^+(t,x)), & (t,x) \in E^+ \cup P^+_0, \\ \widetilde{Y}^-(h^-(t,x)), & (t,x) \in E^- \cup (P^-_0 \setminus P^+_0), \\ 0, & \text{otherwise}. \end{cases} \tag{64}$$

Clearly then $I \in L^\infty_{M_I}(\Omega^T)$ when the initial data $\widetilde{Y}$ is defined recursively by

$$\widetilde{Y}^{\pm}(t,x) := \begin{cases} Y^{\pm}(t,x), & (t,x) \in P^{\pm}_0 \cap L^{\pm}, \\ \widetilde{Y}^+(h^{\mp}(t,x)), & (t,x) \in (P^{\pm}_0 \setminus L^{\pm}) \cap N^{\mp}_0 \cap g^+(P^+_0), \\ \widetilde{Y}^-(h^{\mp}(t,x)), & (t,x) \in (P^{\pm}_0 \setminus L^{\pm}) \cap N^{\mp}_0 \cap g^-(P^-_0), \\ 0, & \text{elsewhere on } \operatorname{cl} J. \end{cases} \tag{65}$$

Step 2: Satisfaction of the transport equation. We now have to show that $F(I,u;\tau) = 0$ for a choice of $\tau \in L^1(\operatorname{Div}^j u)$. So we pick a test function $\varphi \in C^\infty_c(\mathbb{R}^{n+1})$, and observe, first of all, that the definition (64) yields

$$\int_{\Omega^T \setminus (E^+ \cup E^-)} \langle \nabla\varphi, Iu \rangle \, d\mathcal{L}^{n+1} + \int_{\Omega^T \setminus (E^+ \cup E^-)} \varphi I \operatorname{div} u \, d\mathcal{L}^{n+1} = 0. \tag{66}$$

For the remainder of this step of the proof, we study these integrals on E^+ and E^-. To do so, we have to use the C^1 parametrisation of J_0. We therefore choose a side $\sharp \in \{+,-\}$, and let $P \subset P^\sharp_0$ be such that there exists an open set $V \subset \mathbb{R}^m$ and a one-to-one C^1 Lipschitz function $f : V \to P$. We then define

$$X_f(t,\xi) := X^\sharp(t, f(\xi))$$

on

$$G_f := \bigcup \{(t, q^\sharp(t,x)) \times \{\xi\} \mid (t,x) = f(\xi),\ \xi \in V\}.$$

To improve the legibility of forthcoming formulae, we also write $a_f := a^\sharp \circ f$, and $q_f := q^\sharp \circ f$. Observe that $a_f(\xi) = t$ when $f(\xi) = (t,x)$, so, in particular, $X_f(a_f(\xi),\xi) = f(\xi)$ on V. We then set

$$\begin{aligned} A &:= \{(t, X^\sharp(t,(t_0,x_0)) \mid (t_0,x_0) \in P,\ t \in (t_0, q^\sharp(t_0,x_0))\} \\ &= \{(t, X_f(t,\xi)) \mid \xi \in V,\ t \in (a_f(\xi), q_f(\xi))\} \subset E^\sharp. \end{aligned} \tag{67}$$

Next, from, e.g., [6], we find that $\gamma_{(t_0,x_0)}(t)$ depends continuously on the initial data $(t_0,x_0) = f(\xi)$ for $t \in (a_f(\xi), q_f(\xi))$. Therefore, in particular, $X_f(t,\xi')$ for ξ' close to ξ is defined when $t \in (a_f(\xi), q_f(\xi))$. One may then show, following the arguments in the classical case (that we skip; see, e.g., [24] for a general presentation, or [25] for a short proof for the transport equation), that $X_f \in C^1(G_f)$, and

$$\partial_t \mathcal{J}_n[\nabla X_f(t,\cdot)(\xi)] = (\operatorname{div} u)(t, X_f(t,\xi))\ \mathcal{J}_n[\nabla X_f(t,\cdot)(\xi)] \quad \text{on } G_f. \tag{68}$$

Moreover, from (63) we deduce for $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ that

$$\langle (\nabla_{(t,x)}\varphi)(t, X_f(t,\xi)), u(t, X_f(t,\xi))\rangle = \partial_t[\varphi(t, X_f(t,\xi))] \quad \text{on } G_f. \tag{69}$$

An application of the area formula on the transformation $X_f(t,\cdot\,)$ together with (69) now allow us to calculate

$$\begin{aligned}
\int_A \langle \nabla\varphi, Iu\rangle \, d\mathcal{L}^{n+1} &= \int_0^T \int_\Omega (\chi_A \langle \nabla\varphi, u\rangle)(t,x) \; \widetilde{Y}^\sharp(h(t,x))\, dx\, dt \\
&= \int_0^T \int_V (\chi_A \langle \nabla\varphi, u\rangle)(t, X_f(t,\xi)) \; \widetilde{Y}^\sharp(f(\xi)) \; \mathcal{J}_n[\nabla X_f(t,\cdot\,)(\xi)]\, d\xi\, dt \\
&= \int_V \int_{a_f(\xi)}^{q_f(\xi)} \partial_t[\varphi(t, X_f(t,\xi))] \; \mathcal{J}_n[\nabla X_f(t,\cdot\,)(\xi)]\, dt\, \widetilde{Y}^\sharp(f(\xi))\, d\xi.
\end{aligned}$$

Likewise, we deduce

$$\begin{aligned}
\int_A \varphi I \operatorname{div} u \, d\mathcal{L}^{n+1} &= \int_0^T \int_\Omega (\chi_A \varphi \operatorname{div} u)(t,x) \; \widetilde{Y}^\sharp(h(t,x))\, dx\, dt \\
&= \int_V \int_{a_f(\xi)}^{q_f(\xi)} \varphi(t, X_f(t,\xi)) \; (\operatorname{div} u)(t, X_f(t,\xi)) \; \mathcal{J}_n[\nabla X_f(t,\cdot\,)(\xi)]\, dt\, \widetilde{Y}^\sharp(f(\xi))\, d\xi.
\end{aligned}$$

Integration by parts and an application of (68) now establishes

$$\begin{aligned}
&\int_A \langle \nabla\varphi, Iu\rangle \, d\mathcal{L}^{n+1} + \int_A \varphi I \operatorname{div} u \, d\mathcal{L}^{n+1} \\
&\qquad = \int_V \varphi(t, X_f(t,\xi)) \; \mathcal{J}_n[\nabla X_f(t,\cdot\,)(\xi)]\Big|_{t=a_f(\xi)}^{q_f(\xi)} \widetilde{Y}^\sharp(f(\xi))\, d\xi. \\
&\qquad = \int_V \varphi(g^\sharp(f(\xi))) \; \mathcal{J}_n[\nabla X_f(q_f(\xi),\cdot\,)(\xi)]\, \widetilde{Y}^\sharp(f(\xi))\, d\xi \\
&\qquad\qquad - \int_V \varphi(f(\xi)) \; \mathcal{J}_n[\nabla X_f(a_f(\xi),\cdot\,)(\xi)]\, \widetilde{Y}^\sharp(f(\xi))\, d\xi =: O_2 - O_1.
\end{aligned} \tag{70}$$

Since $(t, X_f(t,\xi))$ at $t = q_f(\xi), a_f(\xi)$ is on the the discontinuity set $\operatorname{cl} J$, here $\mathcal{J}_n[\nabla X_f(q_f(\xi),\cdot\,)(\xi)]$ and $\mathcal{J}_n[\nabla X_f(a_f(\xi),\cdot\,)(\xi)]$ have to be understood as traces with respect to time. Indeed, minding (68), we can for any $t_0 \in (a_f(\xi), q_f(\xi))$ write

$$\mathcal{J}_n[\nabla X_f(t,\cdot\,)(\xi)] = \mathcal{J}_n[\nabla X_f(t_0,\cdot\,)(\xi)] + \int_{t_0}^t (\operatorname{div} u)(s, X_f(s,\xi)) \; \mathcal{J}_n[\nabla X_f(s,\cdot\,)(\xi)]\, ds. \tag{71}$$

Observe that it follows from 70 that $\operatorname{Div}(Iu) - I \operatorname{div} u$ is concentrated on J. We however need to show concentration on J_u^{div}, for which we need to compare the partial solutions for varying source sets $P = f(V)$ covering $P_0^\pm$. To do so, we have to calculate the jacobians in the two terms O_1 and O_2. This forms the bulk of the proof of the present proposition.

Regarding O_1, an application of the area formula on the transformation f yields

$$O_1 = \int_P \varphi(t,x) \frac{\mathcal{J}_n[\nabla X_f(t,\cdot\,)(f^{-1}(t,x))]}{\mathcal{J}_n[\nabla f(f^{-1}(t,x))]} \widetilde{Y}^\sharp(t,x)\, d\mathcal{H}^n(t,x). \tag{72}$$

Now, observe that we may write $f(\xi) = h(t_0, X_f(t_0, \xi))$ when $t_0 \in (a_f(\xi), q_f(\xi))$. (Recall that $h^+ = h^-$ in $\Omega^T \setminus \operatorname{cl} J$.) Minding that $X_f(t_0, f^{-1}(t,x)) = X^\sharp(t_0,(t,x))$, we obtain from the definition of $\mathcal{J}_n$ that

$$\mathcal{J}_n[\nabla f(f^{-1}(t,x))] = \mathcal{J}_n[\nabla h(t_0,\cdot\,)(X^\sharp(t_0,(t,x))]\,\mathcal{J}_n[\nabla X_f(t_0,\cdot\,)(f^{-1}(t,x))], \quad ((t,x) \in P),$$

provided that $\nabla h(t_0,\cdot)$ exists at $X(t_0,(t,x))$ on $X_f(t_0,V)$. We claim that this is indeed the case, and show in Step 4 that

$$\lim_{t_0 \searrow t} \mathcal{J}_n[\nabla h(t_0,\cdot\,)(X(t_0,(t,x)))] = 1/|\langle u^\sharp(t,x), \nu_J(t,x)\rangle|, \quad ((t,x) \in P). \tag{73}$$

Minding (71) and that $\sharp\langle u^\sharp, \nu_J\rangle > 0$ on $P \subset P_0^\sharp$, (73) and (72) give

$$O_1 = \sharp \int_P \varphi(t,x)\langle u^\sharp(t,x), \nu_J(t,x)\rangle \widetilde{Y}^\sharp(t,x)\, d\mathcal{H}^n(t,x). \tag{74}$$

Next we study the term O_2 of (70). We now intend to use the area formula on the transformation $g^\sharp \circ f$. It is not, however, generally Lipschitz, as parts of the flow can end up on different surfaces. But consider a point $\xi \in V$ such that $g^\sharp(f(\xi)) \in N_0^\flat$ for some $\flat \in \{+,-\}$, and let $t_0 \in (a_f(\xi), q_f(\xi))$. Then, as discussed in the beginning of the step, $X_f(t_0,\cdot)$ is locally C^1 at ξ, hence locally Lipschitz. Moreover, we will show in Step 4 that

$$\text{The map } g(t_0,\cdot) \text{ is locally } C^1 \text{ at } X_f(t_0,\xi) \text{ when } t_0 \text{ and } \xi \text{ are as above.} \tag{75}$$

Thus $g^\sharp \circ f = g(t_0, X_f(t_0,\cdot))$ is locally Lipschitz at such $\xi \in V$. From the uniqueness of solutions γ, discussed before (60)–(62), it follows that $g^\sharp|P_0^\sharp$ is one-to-one in a neighbourhood of $f(\xi)$. This allows us to apply the Vitali covering theorem on V to yield a disjoint family $\{V^i\}_{i=0}^\infty$ of open balls such that $\mathcal{L}^n(V \setminus \bigcup_i V^i) = 0$, and where $g^\sharp|P^i$ is a one-to-one (Lipschitz) map with inverse $h^{\flat_i}$ between $P^i := f(V^i)$ and $N^i := g(P^i) \subset N^{\flat_i}$.

It now follows that

$$O_2 = \sum_i O_2^i := \sum_i \int_{V^i} \varphi(g^\sharp(f(\xi)))\; \mathcal{J}_n[\nabla X_f(q_f(\xi),\cdot\,)(\xi)]\, \widetilde{Y}^\sharp(f(\xi))\, d\xi$$

Similarly to (74), an application of the area formula on the transformation $g^\sharp \circ f$ now yields

$$O_2^i = \int_{N^i} \varphi(t,x) \frac{\mathcal{J}_n[\nabla X_f(t,\cdot\,)((f^{-1} \circ h^{\flat_i})(t,x)))]}{\mathcal{J}_n[\nabla (g^\sharp \circ f)((f^{-1} \circ h^{\flat_i})(t,x))]} \widetilde{Y}^\sharp(h^{\flat_i}(t,x))\, d\mathcal{H}^n(t,x).$$

Writing $(g^\sharp \circ f)(\xi) = g(t_0, X_f(t_0,\xi))$ when $t_0 \in (a_f(\xi), q_f(\xi))$, yields again

$$\begin{aligned}&\mathcal{J}_n[\nabla (g^\sharp \circ f)((f^{-1} \circ h^{\flat_i})(t,x))] \\ &\qquad = \mathcal{J}_n[\nabla g(t_0,\cdot)(X^\sharp(t_0, h^{\flat_i}(t,x)))]\,\mathcal{J}_n[\nabla X_f(t_0,\cdot\,)((f^{-1} \circ h^{\flat_i})(t,x))], \quad ((t,x) \in N^i),\end{aligned}$$

provided that $\nabla g(t_0,\cdot)(X^\sharp(t_0, h^{\flat_i}(t,x)))$ exists. Again, we claim that this is the case, and

$$\lim_{t_0 \nearrow t} \mathcal{J}_n[\nabla g(t_0,\cdot)(X^\sharp(t_0, h^{\flat_i}(t,x)))] = 1/|\langle u^{\flat_i}(t,x), \nu_J(t,x)\rangle|, \quad ((t,x) \in N^i). \tag{76}$$

Minding that $\flat_i \langle u^{\flat_i}, \nu_J \rangle < 0$ on $N^i \subset N_0^{\flat_i}$, it follows that

$$
\begin{aligned}
O_2 &= \sum_i \int_{N^i} \varphi(t,x)(-\flat_i)\langle u^{\flat_i}(t,x), \nu_J(t,x)\rangle \widetilde{Y}^\sharp(h^{\flat_i}(t,x))\, d\mathcal{H}^n(t,x) \\
&= \sum_{\flat\in\{+,-\}} \flat \int_{g^\sharp(P)\cap N_0^\flat} \varphi(t,x)\langle u^\flat(t,x), -\nu_J(t,x)\rangle \widetilde{Y}^\sharp(h^\flat(t,x))\, d\mathcal{H}^n(t,x).
\end{aligned} \tag{77}
$$

Provided that (73) and (76) hold along with (75), it follows from plugging (74) and (77) into (70) that

$$
\begin{aligned}
&\int_A \langle \nabla\varphi, Iu\rangle\, d\mathcal{L}^{n+1} + \int_A \varphi I \operatorname{div} u\, d\mathcal{L}^{n+1} \\
&\qquad = -\Big(\sharp \int_P \varphi \widetilde{Y}^\sharp \langle u^\sharp, \nu_J\rangle\, d\mathcal{H}^n + \sum_{\flat\in\{+,-\}} \flat \int_{N_0^\flat\cap g(P)} \varphi(\widetilde{Y}^\sharp \circ h^\flat)\langle u^\flat, \nu_J\rangle\, d\mathcal{H}^n\Big).
\end{aligned} \tag{78}
$$

Now, observe that the Vitali covering theorem again provides us with a disjoint family of sets $\{P^i\}_{i=0}^\infty$ such that $\mathcal{H}^n(P_0^\sharp \setminus \bigcup_i P^i) = 0$, and there exist open sets $V^i \subset \mathbb{R}^m$ and one-to-one C^1 Lipschitz maps $f^i : V^i \to P^i$. The corresponding sets A^i defined by (67) then cover almost all of $E^\sharp$, due to the uniqueness of solution curves γ on $P_0^\sharp$. Therefore, recalling (66), we may deduce from (78) that

$$
\begin{aligned}
K &:= -\Big(\int_\Omega \langle \nabla\varphi, Iu\rangle\, d\mathcal{L}^{n+1} + \int_\Omega \varphi I \operatorname{div} u\, d\mathcal{L}^{n+1}\Big) \\
&\qquad = \sum_{\sharp\in\{+,-\}} \Big(\sharp \int_{P_0^\sharp} \varphi \widetilde{Y}^\sharp \langle u^\sharp, \nu_J\rangle\, d\mathcal{H}^n + \sum_{\flat\in\{+,-\}} \flat \int_{N_0^\flat\cap g(P_0^\sharp)} \varphi(\widetilde{Y}^\sharp \circ h^\flat)\langle u^\flat, \nu_J\rangle\, d\mathcal{H}^n.\Big)
\end{aligned}
$$

Exchanging orders of the sums on the second term, we get

$$
K = \sum_{\sharp\in\{+,-\}} \sharp\Big(\int_{P_0^\sharp} \varphi \widetilde{Y}^\sharp \langle u^\sharp, \nu_J\rangle\, d\mathcal{H}^n - \sum_{\flat\in\{+,-\}} \flat \int_{N_0^{-\sharp}\cap g(P_0^\flat)} \varphi(\widetilde{Y}^\flat \circ h^{-\sharp})\langle u^{-\sharp}, \nu_J\rangle\, d\mathcal{H}^n\Big).
$$

By an application of (65) we may restrict attention to $L^\pm = P_u^\pm \cap J_u^{\mathrm{div}}$, yielding

$$
\begin{aligned}
K &= \sum_{\sharp\in\{+,-\}} \sharp\Big(\int_{P_0^\sharp\cap L^\sharp} \varphi Y^\sharp \langle u^\sharp, \nu_J\rangle\, d\mathcal{H}^n - \sum_{\flat\in\{+,-\}} \flat \int_{g(P_0^\flat)\cap L^\sharp\cap P_0^\sharp} \varphi(\widetilde{Y}^\flat \circ h^{-\sharp})\langle u^{-\sharp}, \nu_J\rangle\, d\mathcal{H}^n\Big) \\
&\qquad = \int_{J_u^{\mathrm{div}}} \varphi \sum_{\sharp\in\{+,-\}} \sharp\Big(\chi_{P_0^\sharp} Y^\sharp \langle u^\sharp, \nu_J\rangle - \sum_{\flat\in\{+,-\}} \flat \chi_{g(P_0^\flat)\cap P_0^\sharp} (\widetilde{Y}^\flat \circ h^{-\sharp})\langle u^{-\sharp}, \nu_J\rangle\Big)\, d\mathcal{H}^n.
\end{aligned}
$$

Since $Y^\pm$ and u are bounded and $\mathcal{H}^n(J) < \infty$, we deduce that $K = \int \varphi\tau\, d\operatorname{Div}^j u$ for some $\tau \in L^1(\operatorname{Div}^j u)$ (independent of φ). This shows $F(I, u; \tau) = 0$.

Step 3: Traces. Let $(t,x) \in P_0^+$ and consider a small neighbourhood $U := B((t,x),\delta)$ as in the beginning of the proposition, split into open halves $U^+ := U^+(t,x)$ and $U^- := U^-(t,x)$ on different sides of $\operatorname{cl} J \cap U$. Let us set $w := u\chi_{U^+}$. If we repeat Step 2 with I and w instead of u, and a

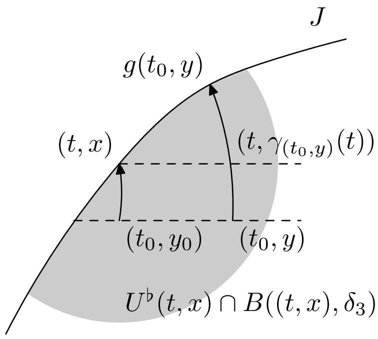

Figure 3: The situation in the beginning of Step 4 of Proposition 2.

test function $\varphi \in C_c^\infty(U)$, the term O_2 will be zero. We may, in fact, assume that $\delta > 0$ is small enough that $\operatorname{cl} J \cap U$ is the image P of a single C^1 map f, and $U^+ \subset A$. From (74) and (70), we then obtain

$$\int_U \langle \nabla\varphi, Iw\rangle \, d\mathcal{L}^{n+1} + \int_U \varphi I \operatorname{div} w \, d\mathcal{L}^{n+1} = -\int_{\operatorname{cl} J\cap U} \varphi \widetilde{Y}^+ \langle u^+, \nu_J\rangle \, d\mathcal{H}^n,$$

or

$$\int_{U^+} \langle \nabla\varphi, Iu\rangle \, d\mathcal{L}^{n+1} + \int_{U^+} \varphi I \operatorname{div} u \, d\mathcal{L}^{n+1} = -\int_{\operatorname{cl} J\cap U} \varphi \widetilde{Y}^+ \langle u^+, \nu_J\rangle \, d\mathcal{H}^n.$$

This shows that $\operatorname{Div} Iu \llcorner U^+ = I \operatorname{div} u \mathcal{L}^{n+1}$, because $\operatorname{cl} J \cap U$ does not intersect U^+. Hence the normal trace of Iu on the boundary ∂U^+ satisfies

$$\operatorname{Tr}(Iu, \partial U^+)(\varphi) = \int_{U^+} \langle \nabla\varphi, Iu\rangle \, d\mathcal{L}^{n+1} + \int_{U^+} \varphi \, d \operatorname{Div}(Iu) = -\int_{\operatorname{cl} J\cap U} \varphi \widetilde{Y}^+ \langle u^+, \nu_J\rangle \, d\mathcal{H}^n.$$

Consequently, we deduce that the normal trace on the positive side of P_0^+, which is in the interior of U^+, satisfies

$$\operatorname{Tr}^+(Iu, P_0^+) = \widetilde{Y}^+ \langle u^+, \nu_J\rangle \mathcal{H}^n \llcorner P_0^+.$$

But Proposition 3 shows that $\operatorname{Tr}^+(Iu, P_0^+) = I^+ \langle u^+, \nu_J\rangle \mathcal{H}^n \llcorner P_0^+$. Since $\langle u^+, \nu_J\rangle > 0$ on P_0^+, it follows that $I^+ = \widetilde{Y}^+$ on P_0^+. In particular, since $L^+ \setminus P_0^+$ is $\mathcal{H}^n$-negligible, we have $I^+ = Y^+$ a.e. on L^+. This is what we had to show.

Repeating the arguments above on the "minus side" U^- of P_0^- yields $I^- = Y^-$ on L^-, showing that I satisfies the trace claim of the proposition.

Step 4: Differentiability properties of g and h. To complete the proof of the present proposition, it remains to show the Jacobian formulae (73) and (76) along with (75). As the proof of (73) is analogous to that of (76), merely traversing the flow backwards, we only show the latter. See also [6] for other considerations of similar nature.

Let then $(t, x) \in N_0^\flat$, and $(t_0, y_0) \in \Omega^T \setminus \operatorname{cl} J$ with $t_0 < t$ be such that $(t, x) = g(t_0, y_0)$. Denote $\bar{u} := u(t, x)$, and $\bar{\nu} := \nu_{J_u}(t, x)$. Let V_0 be an open neighbourhood of y_0 such that $t_0 \in (a(t_0, y), q(t_0, y))$ when $y \in V_0$. (Such a neighbourhood exists, as discussed in Step 2.) Also write $t_y = q(t_0, y)$ and $x_y = g(t_0, y)$, and set $V_0^\uparrow := \{y \in V_0 \mid t_y \geq t\}$, and $V_0^\downarrow := V_0 \setminus V_0^\uparrow$. For $y \in V_0^\uparrow$, we get (see Figure 3)

$$g(t_0, y) - g(t_0, y_0) = g(t, \gamma_{(t_0,y)}(t)) - (t, x), \tag{79}$$

where we have to mind the correct side $\flat$ at (t,x). Given $\epsilon > 0$, we will show below that there exist $\delta_3 > 0$ and $\epsilon' \in (0,\epsilon)$ such that when $(t,z) \in U^\flat(t,x) \cap B((t,x),\delta_3)$, we can (for fixed z) write

$$g(t,z) - (t,x) = \left(\mathrm{id} - \frac{\widetilde{u} \otimes \widetilde{\nu}}{\langle \widetilde{u}, \widetilde{\nu}\rangle}\right)(0, z-x) \tag{80}$$

for some

$$\widetilde{u} \in B(\bar{u}, \epsilon'), \quad \text{and} \quad \widetilde{\nu} \in B(\bar{\nu}, \epsilon'),\ \|\widetilde{\nu}\| = 1, \tag{81}$$

all of which satisfy

$$\left\| \frac{\bar{u} \otimes \bar{\nu}}{\langle \bar{u}, \bar{\nu}\rangle} - \frac{\widetilde{u} \otimes \widetilde{\nu}}{\langle \widetilde{u}, \widetilde{\nu}\rangle} \right\| \le \epsilon. \tag{82}$$

We may also write

$$\gamma_{(t_0,y)}(t) - x = d_0(y) := (y - y_0) + \int_{t_0}^{\min\{t,t_y\}} u(s, \gamma_{(t_0,y)}(s)) - u(s, \gamma_{(t_0,y_0)}(s))\, ds. \tag{83}$$

On the other hand, for $y \in V_0^{\downarrow}$, we may similarly to (79) write

$$g(t_0,y) - g(t_0,y_0) = (t_y, x_y) - g^\flat(t_y, \gamma_{(t_0,y_0)}(t_y)). \tag{84}$$

Also analogously to (80), it can be shown that there exists $\delta_4 > 0$ such that whenever $(t_x, x_y), (t_x, z) \in U^\flat(t,x) \cap B((t,x),\delta_4)$, we have

$$(t_y, x_y) - g^\flat(t_y, z) = \left(\mathrm{id} - \frac{\widetilde{u} \otimes \widetilde{\nu}}{\langle \widetilde{u}, \widetilde{\nu}\rangle}\right)(0, x_y - z) \tag{85}$$

for some $\widetilde{u}$ and $\widetilde{\nu}$ satisfying (81) and (82). Observing that also

$$x_y - \gamma_{(t_0,y_0)}(t_y) = d_0(y),$$

it follows from (79), (80) and (84), (85) that in some open neighbourhood $V \subset V_0$ of y_0, we have

$$g(t_0,y) - g(t_0,y_0) = \left(\mathrm{id} - \frac{\widetilde{u} \otimes \widetilde{\nu}}{\langle \widetilde{u}, \widetilde{\nu}\rangle}\right)(0, d_0(y)) \tag{86}$$

for some $\widetilde{u}$ and $\widetilde{\nu}$ dependent on y and satisfying (81) and (82). Since $\epsilon > 0$ was arbitrary, it easily follows, using (82), that $g(t_0,\cdot)$ is continuous at y_0. In particular $q(t_0,\cdot) = \langle (1,0), g(t_0,\cdot)\rangle$ is continuous at y_0. By repeating the arguments above with other $y_0 \in V$, we obtain continuity on V.

To show differentiability, mind that, by assumption, u is smooth in $\Omega^T \setminus \mathrm{cl}\, J$. Moreover, since $\gamma_{(t_0,y_0)}(s) \notin \mathrm{cl}\, J$, by classical results, $y \mapsto \gamma_{(t_0,y)}(s)$ is locally Lipschitz and C^1 for $s \in (t_0, t)$ (again, similarly to as discussed in Step 2). By the continuity of $t_y = q(t_0, y)$ on V, shown above, it thus follows that d_0 is C^1 on a neighbourhood $V' \subset V$ of y_0. Since $\epsilon > 0$ was arbitrary, it is then easy to deduce from (86), using (82), that

$$\nabla g(t_0,\cdot)(y_0) = \bar{H}\nabla d_0(y_0), \quad \text{where } \bar{H}(v) := \left(\mathrm{id} - \frac{\bar{u} \otimes \bar{\nu}}{\langle \bar{u}, \bar{\nu}\rangle}\right)(0, v).$$

By the already observed continuity of $g(t_0,\cdot)$ on V, we deduce that $\bar u = u(g(t_0,y_0))$ and $\bar v = \nu_{J_u}(g(t_0,y_0))$ depend continuously on y_0 in V. We can therefore conclude that $g(t_0,\cdot)$ is locally C^1, so (75) holds. Moreover, by application of some elementary row transformations and the Cauchy-Binet formula, one can show that $\mathcal{J}_n[\bar H] = 1/|\langle \bar u, \bar \nu\rangle|$. Therefore, observing that $\lim_{t_0 \nearrow t} \nabla d_0(y_0) = \mathrm{id}$ (where y_0 varies with t_0, converging to x), we obtain (76).

To complete the proof, we now have to show (80). Since the proof will be of local nature, to ease the notation, we translate the problem so that $(t,x)=0$. Since $0=(t,x) \in N' \subset N_0^\flat$, we may assume that $(0,y) \in U^\flat(0) \subset B(0,\delta)$, where δ is as in the beginning of the proposition. Let us also observe that

$$g^\flat(0,y) = (0,y) + \int_0^{q^\flat(0,y)} u(s,\gamma_{(0,y)}(s))\,ds. \tag{87}$$

Let then $\epsilon > 0$ be arbitrary. Since $\langle \bar u, \bar \nu\rangle \neq 0$, there exists $\epsilon' \in (0,\epsilon)$ such that (82) holds whenever (81) does. There also exists $\delta_1 \in (0,\delta)$ such that $\|u(s,y)-\bar u\| \le \epsilon'$ when $(s,y) \in U^\flat(0) \cap B(0,\delta_1)$. Moreover, there exists $\delta_2 \in (0,\delta_1)$ such that

$$\operatorname{cl} J \cap B(0,\delta_2) \subset K_{\epsilon'} := \bigcup \{\widetilde\nu^\perp \mid \widetilde \nu \in B(\bar\nu,\epsilon'),\ \|\widetilde\nu\|=1\}.$$

Let us abbreviate $q(y) := q^\flat(0,y)$. Applying (87), we now have

$$g^\flat(0,y) = (0,y) + q(y)\widetilde u \quad \text{for some } \widetilde u \in B(\bar u,\epsilon'), \tag{88}$$

as long as we have $(s,\gamma_{(0,y]}(s)) \in U_\flat(0) \cap B(0,\delta_1)$ for $s \in [0,q(y))$. But this follows if $q(y)$ is small enough that

$$(0,y) + q(y)\widetilde u \in B(0,\delta_1) \quad \text{for all } \widetilde u \in B(\bar u,\epsilon'). \tag{89}$$

To find $q(y)$, we want to solve $(0,y)+q(y)\widetilde u \in \operatorname{cl} J$. Approximating $\operatorname{cl} J$ by K_ϵ, we have

$$(0,y) + q(y)\widetilde u \in \widetilde \nu^\perp \quad \text{for some } \widetilde\nu \in B(\bar\nu,\epsilon),\ \|\widetilde\nu\|=1.$$

Taking the inner product on both sides by $\widetilde\nu$, we obtain

$$q(y) = -\langle (0,y),\widetilde\nu\rangle / \langle \widetilde u,\widetilde \nu\rangle. \tag{90}$$

This is well-defined thanks to (82). Inserting $q(y)$ into the condition (89), it becomes

$$\left(\mathrm{id} - \frac{\widetilde u \otimes \widetilde \nu}{\langle \widetilde u,\widetilde\nu\rangle}\right)(0,y) \in B(0,\delta_1).$$

By choosing $\delta_3 \in (0,\delta_2)$ small enough, by (82), this can be made to hold for all $(0,y) \in B(0,\delta_3)$ and $\widetilde\nu$ and $\widetilde u$ satisfying (81). In consequence, (88) holds for $(0,y) \in B(0,\delta_3)$. Minding the expression (90) for $q(y)$, and the translation of (t,x) to 0, this establishes (80), thus completing the proof. □

We next state our main existence theorem. For the stronger version of it, bounding τ, we assume boundedness on J_u^{div} from

$$\kappa_u(x) := \frac{|\langle u^+(x),\nu_{J_u}(x)\rangle| + |\langle u^-(x),\nu_{J_u}(x)\rangle|}{|\langle u^+(x),\nu_{J_u}(x)\rangle - \langle u^-(x),\nu_{J_u}(x)\rangle|}. \tag{91}$$

What this roughly says is that if $\operatorname{Div}^j u$ approaches zero on J_u^{div}, then the normal traces $\langle u^\pm,\nu_{J_u}\rangle$ must also approach zero at a similar rate.

Theorem 5. *Suppose $u \in X_u$ with $\mathcal{E}u \in L^2(\Omega^T;\mathbb{R}^{(n+1)\times(n+1)})$ and $\mathcal{H}^{m-1}(J_u) < \infty$. Given $Y^\pm \in L^\infty_{M_I}(L_u^\pm)$ for some $M_I \geq 0$, there then exist $I \in X_I$ and $\tau \in L^1(\mathrm{Div}^j u)$ with $F(I,u;\tau) = 0$ and*

$$\tau \,\mathrm{Div}^j u = \langle \widetilde{Y}^+ u^+ - \widetilde{Y}^- u^-, \nu_{J_u}\rangle \mathcal{H}^n \llcorner J_u \tag{92}$$

for some $\widetilde{Y}^\pm \in L^\infty_{M_I}(J_u)$ satisfying $\widetilde{Y}^\pm = Y^\pm$ on $L_u^\pm$ (a.e.).

Additionally, if for some $M \geq 1$ we have $\kappa_u \in L^\infty_M(J_u^{\mathrm{div}})$, then $\tau \in L^\infty_{M_I M}(J_u^{\mathrm{div}})$.

Proof. We take an approximating sequence $\{u^i\}_{i=0}^\infty \subset \mathcal{W}^\infty(\Omega^T) \cap L^\infty_{M_u}(\Omega^T;\mathbb{R}^{n+1})$ of u, as given by Theorem 3, extending u^i outside Ω^T by zero. We then have

$$u^i \to u \text{ strongly in } L^2(\Omega;\mathbb{R}^{n+1}), \tag{93}$$

$$\mathcal{E}u^i \to \mathcal{E}u \text{ strongly in } L^2(\Omega;\mathbb{R}^{(n+1)\times(n+1)}), \tag{94}$$

$$\|(u^i)^\pm - u^\pm\|_{L^1(J_u \cup J_{u^i};\mathbb{R}^n)} \to 0, \text{ and} \tag{95}$$

$$\mathcal{H}^n(J_{u^i} \Delta J_u) \to 0. \tag{96}$$

Observe, moreover, from the finite element construction in the proof of Theorem 3 (Lemma 2), that the $(1,b)$ structure of u is preserved, i.e., $u^i \in X_u$.

Next we construct a solution to the (extended) transport equation with velocity field u^i. For initial/source data, we set

$$(Y^i)^\pm := Y^\pm \quad \text{on} \quad (L^i)^\pm := L^\pm_{u^i} \cap L^\pm_u. \tag{97}$$

Proposition 2 then gives a solution pair $I^i \in X_I$ and $\tau^i \in L^1(\mathrm{Div}^j u^i)$ to $F(I^i, u^i; \tau^i) = 0$ with $(I^i)^\pm = (Y^i)^\pm$ on $(L^i)^\pm$, and $(I^i)^+ = (I^i)^-$ on $J_{u^i} \setminus L^i$, where we denote $L^i := (L^i)^+ \cup (L^i)^-$. For later use, we also introduce the analogous notation $L_u := L_u^+ \cup L_u^-$.

We cannot use Theorem 4 as this point, because $\{\tau^i\}_{i=0}^\infty$ may not be bounded in L^∞, and because $\{I^i\}_{i=0}^\infty$ may not converge pointwise-a.e. The sequence $\{I^i\}_{i=0}^\infty$ however is bounded in $L^\infty(\Omega^T)$, so we may assume it weak* convergent to some $I \in L^\infty(\Omega^T)$. Applying (93) and (94), it therefore follows for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ that

$$-\int_{\mathbb{R}^{n+1}} \langle \nabla\varphi, I^i u^i\rangle \,dx - \int_{\mathbb{R}^{n+1}} \varphi I^i \,\mathrm{div}\, u^i \,dx \to -\int_{\mathbb{R}^{n+1}} \langle \nabla\varphi, Iu\rangle \,dx - \int_{\mathbb{R}^{n+1}} \varphi I \,\mathrm{div}\, u \,dx. \tag{98}$$

It remains to study the behaviour of the term $\tau^i \,\mathrm{Div}^j u^i$ of $F(I^i, u^i; \tau^i)$. By Proposition 3, we have

$$\tau^i \,\mathrm{Div}^j u^i = \mathrm{Div}^j(I^i u^i) = \langle (I^i)^+(u^i)^+ - (I^i)^-(u^i)^-, \nu_{J_{u^i}}\rangle \mathcal{H}^n \llcorner J_{u^i}, \tag{99}$$

where $(I^i)^\pm$ exist on J_{u^i} when $\langle (u^i)^\pm, \nu_{J_{u^i}}\rangle \neq 0$, and are defined arbitrarily otherwise. Minding that $I^i \in L^\infty_{M_I}(\Omega^T)$, we extend $(I^i)^\pm$ to $L^\infty_{M_I}(J_u \cup J_{u^i})$ by defining $(I^i)^\pm = 0$ on $J_u \setminus J_{u^i}$. After possibly switching to subsequences, unrelabelled, we may then assume the sequences $\{(I^i)^\pm | J_u\}_{i=0}^\infty$ convergent weakly* in $L^\infty(J_u)$ to some $\widetilde{Y}^\pm \in L^\infty_{M_I}(J_u)$. Moreover, by application of Lemma 7 in the Appendix (with $A = L_u^\pm$, $\mu = \mathcal{H}^n \llcorner L_u^\pm$, and $v^i = \min\{\max\{0, \langle (u^i)^\pm, \nu_{J_u}\rangle\}, |\langle (u^i)^\pm - (u^i)^\pm, \nu_{J_u}\rangle|\}$), we obtain

$$\mathcal{H}^n(L_u^\pm \setminus (L^i)^\pm) = \mathcal{H}^n(L_u^\pm \setminus L^\pm_{u^i}) \to 0, \quad (i \to \infty). \tag{100}$$

(The converse, $\mathcal{H}^n(L^\pm_{u^i} \setminus L^\pm_u) \to 0$, may not hold.) Therefore, minding that $(I^i)^\pm = (Y^i)^\pm = Y^\pm$ on $(L^i)^\pm$, we deduce that $\widetilde{Y}^\pm = Y^\pm$ on $L_u^\pm$, as required by the lemma.

Employing (95), we now find (see, e.g., [21])

$$(I^i)^\pm (u^i)^\pm \rightharpoonup \widetilde{Y}^\pm u^\pm \quad \text{weakly in } L^1(J_u;\mathbb{R}^{n+1}), \quad (i\to\infty). \tag{101}$$

Since, by (96), $\mathcal{H}^n(J_u^i \Delta J_u)\to 0$, and I^i and u^i are bounded, we deduce for all $\varphi\in C_c(\mathbb{R}^{n+1})$ that

$$\begin{aligned}\int \varphi\tau^i \, d\operatorname{Div}^j u^i &= \int_{J_{u^i}} \varphi\big\langle (I^i)^+(u^i)^+ - (I^i)^-(u^i)^-, \nu_{J_{u^i}}\big\rangle \, d\mathcal{H}^n \\ &\to \int_{J_u} \varphi\big\langle \widetilde{Y}^+u^+ - \widetilde{Y}^-u^-, \nu_{J_u}\big\rangle \, d\mathcal{H}^n.\end{aligned} \tag{102}$$

To see this, observe that the integral on the left may be written as the sum of integrals over J_u and $J_{u^i}\setminus J_u$, due to the extension of $(I^i)^\pm$ to $J_u\setminus J_{u^i}$ by zero.

Next, we observe that by our choice (97) of $(L^i)^\pm$ and $(Y^i)^\pm$, we may refine (99) into

$$\tau^i \operatorname{Div}^j u^i = \big\langle (I^i)^+(u^i)^+ - (I^i)^-(u^i)^-, \nu_{J_{u^i}}\big\rangle \mathcal{H}^n \llcorner L^i + \widetilde{I}^i \operatorname{Div}^j u^i \llcorner (J_{u^i}\setminus L^i), \tag{103}$$

for some $\widetilde{I}^i \in L^\infty_{M_I}(J_{u^i}\setminus L^i)$. Calculating that

$$(J_{u^i}\setminus L^i)\Delta(J_u\setminus L_u) \subset (J_u\Delta J_{u^i}) \cup (L_u\setminus L^i),$$

due to the choice (97) of $L^i\subset L_u$, we deduce from (96) and (100) that

$$\mathcal{H}^n((J_{u^i}\setminus L^i)\Delta(J_u\setminus L_u)) \to 0.$$

By this and (95), it follows that

$$\operatorname{Div}^j u^i \llcorner (J_{u^i}\setminus L^i) \to \operatorname{Div}^j u \llcorner (J_u\setminus L_u) \quad \text{in total variation}, \quad (i\to\infty).$$

Because $\widetilde{I}^i \in L^\infty_{M_I}(J_{u^i}\setminus L^i)$, following the proof of Theorem 4, we then observe the existence of $\tau_1 \in L^\infty_{M_I}(J_u\setminus L_u)$ such that for an unrelabelled subsequence

$$\widetilde{I}^i \operatorname{Div}^j u^i \llcorner (J_{u^i}\setminus L^i) \overset{*}{\rightharpoonup} \tau_1 \operatorname{Div}^j u \llcorner (J_u\setminus L_u) \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^{n+1}), \quad (i\to\infty), \tag{104}$$

We also have $\tau_1\in L^1(\operatorname{Div}^j u)$, because, by the assumption $P(u)<\infty$, we have $\mathcal{H}^n(J_u)<\infty$.

Regarding the first term on the right side of (103), we deduce from (101) and (100) that, again after possibly moving to an unrelabelled subsequence

$$\big\langle (I^i)^+(u^i)^+ - (I^i)^-(u^i)^-, \nu_{J_{u^i}}\big\rangle \mathcal{H}^n \llcorner L^i \overset{*}{\rightharpoonup} \big\langle \widetilde{Y}^+u^+ - \widetilde{Y}^-u^-, \nu_{J_u}\big\rangle \mathcal{H}^n \llcorner L_u, \quad (i\to\infty), \tag{105}$$

weakly* in $\mathcal{M}(\mathbb{R}^{n+1})$. Recalling that $L_u\subset J_u^{\mathrm{div}}$, we may write

$$\big\langle \widetilde{Y}^+u^+ - \widetilde{Y}^-u^-, \nu_{J_u}\big\rangle \mathcal{H}^n \llcorner L_u = \tau_2 \operatorname{Div}^j u \tag{106}$$

for some Borel function τ_2. In fact, since the assumption $P(u)<\infty$ implies $\mathcal{H}^n(J_u)<\infty$, and because both $u^\pm$ and $\widetilde{Y}^\pm$ are bounded, we may conclude that $\tau_2\in L^1(\operatorname{Div}^j u)$.

Let us now set $\tau := \tau_1+\tau_2$. Then $\tau\in L^1(\operatorname{Div}^j u)$, and by combining the observations (103)–(106), we find for all $\varphi\in C_c(\mathbb{R}^{n+1})$ that

$$\int \varphi\tau^i \, d\operatorname{Div}^j u^i \to \int \varphi\tau \, d\operatorname{Div}^j u. \tag{107}$$

Comparing (102) to (107), we deduce that (92) holds. Moreover, since $F(I^i, u^i; \tau^i) = 0$, it follows from (98) and (107) that $F(I, u; \tau) = 0$.

We still have to consider the case $\kappa_u \in L^\infty_M(J_u^{\mathrm{div}})$. We already showed that $\tau_1 \in L^\infty_{M_I}(J_u \setminus L_u)$. For τ_2, we may deduce from (106) that $|\tau_2(x)| \leq M_I|\kappa_u(x)|$ for $x \in L_u$. Hence $\|\tau_2\|_{L^\infty(L_u)} \leq M_I M$, and so it follows that $\tau \in L^\infty_{M_I M}(J_u^{\mathrm{div}})$, as claimed. The proof can now be concluded. □

Remark 4. We have not shown that the traces $I^\pm$ would equal $Y^\pm$ on $L_u^\pm$, only that τ is of a form that would be had if this were the case. From the construction it is apparent that if we had the strict convergence $\|I - I^i\|_{L^1(\Omega^T)} + ||DI|(\Omega^T) - |DI^i|(\Omega^T)| \to 0$, in which case traces are convergent, then this property would hold. Proposition 3 shows that the one one-sided Lebesgue limits $I^\pm$ however exist on $N_u^\pm \cup P_u^\pm$, and $(Iu)^\pm = I^\pm u^\pm$. Thus, in particular, $\langle I^+u^+ - I^-u^-, \nu_{J_u}\rangle = \langle \widetilde{Y}^+u^+ - \widetilde{Y}^-u^-, \nu_{J_u}\rangle$. From this it follows that $I^\pm = \widetilde{Y}^\pm$ when $\langle u^\mp, \nu_{J_u}\rangle = 0$, and so the trace is as requested, e.g., at the initial time $t = 0$.

Remark 5. One further remark is in order, regarding the stability of the condition $0 \in F(I, u)$. All three, Theorem 4, Proposition 1, and the proof of Theorem 5 provide a stability result of one type or the other. Theorem 4 is the strongest in the sense that the jump sets of u^i may vary, but in no way does it show the convergence of the traces of I^i on the jump parts $L_u^\pm$ of the jump set. Proposition 1 provides a stability result that is much stronger with regard to initial conditions, but only for mollifier approximations of u. Finally, the proof of Theorem 5 provides a stability result with regard to the relatively strong form of convergence (93)–(96). It still shows full stability with regard to initial data, because at time zero $\langle u^-, \nu_{J_u}\rangle = 0$, but for sources on jumps in space-time not satisfying a property of this type, the stability is somewhat weaker.

A limitation with the stability result in the proof of Theorem 5 is that the jump set is expected to be mostly stationary. To overcome this, and to support more arbitrary approximating sequences $\{u^i\}_{i=0}^\infty$, the techniques of the outer-semicontinuity proof of Theorem 4 and of Theorem 5 could be combined. For example, by requiring that each $\kappa_{u^i} \in L^\infty_M(J_{u^i}^{\mathrm{div}})$, so that τ^i are also bounded, we could get (107) by using the techniques of Theorem 4, even when the jump sets J_{u^i} are not mostly stationary. To get (102) in this case, we could require in advance

1. The weak* convergence of $(Y^i)^\pm\langle (u^i)^\pm, \nu_{J_{u^i}}\rangle \mathcal{H}^n \llcorner L_{u^i}^\pm$ to $Y^\pm\langle u^\pm, \nu_{J_u}\rangle\mathcal{H}^n \llcorner L_u^\pm$, and
2. Weak* convergence of $\langle (u^i)^\pm, \nu_{J_{u^i}}\rangle\mathcal{H}^n \llcorner (J_{u^i}^{\mathrm{div}} \setminus L_{u^i}^\pm)$ to $\langle u^\pm, \nu_{J_u}\rangle\mathcal{H}^n \llcorner (J_u^{\mathrm{div}} \setminus L_u^\pm)$, along with convergence of total variations.

Following the techniques of the outer-semicontinuity proof in Theorem 4 again, the latter condition would then show the weak* convergence of a subsequence of $(I^i)^\pm\langle (u^i)^\pm, \nu_{J_{u^i}}\rangle\mathcal{H}^n \llcorner (J_{u^i}^{\mathrm{div}} \setminus L_{u^i}^\pm)$ to some $(\widetilde{Y})^\pm\langle u^\pm, \nu_{J_u}\rangle\mathcal{H}^n \llcorner (J_u^{\mathrm{div}} \setminus L_u^\pm)$. Hence, by combining with the first condition, we would obtain (102). Again comparing to (107) would then show stability of solutions in the weak sense (92).

3.5. Renormalisation and uniqueness

We finally study the uniqueness of solutions I to $0 \in F(I, u)$ subject to one-sided traces on $L_u^\pm$. (At this point it is advisable to recall the definition of these sets from (58), as well as that of J_u^{div}.) We begin by rewriting the condition $F(I, u; \tau) = 0$ with respect to integral over time. A Gronwall-type estimate then leads to a preliminary uniqueness result under positivity assumptions on I and the bound $\int_0^T \|\max\{0, \operatorname{div} b(t, \cdot)\}\|_{L^\infty(\Omega)}\, dt < \infty$. This bound is akin to what is found in other recent works [4, 5], although by $\operatorname{div} b(t, \cdot)$ we refer to the mere absolutely continuous part of the distributional divergence $\operatorname{Div} b(t, \cdot)$. Finally, we do away with the positivity assumption with the help of a renormalisation argument.

Lemma 4. *Let $I \in X_I$, $u \in X_u$, and $\tau \in L^1(\operatorname{Div}^j u)$ with $F(I, u; \tau) = 0$. Denote $I_t(x) := I(t, x)$, and $b_t(x) := b(t, x)$, where $u = (1, b)$. Then for all $\theta \in C_c^\infty(\mathbb{R})$,*

$$\begin{aligned} -\int_0^T \theta'(t) \left[\int_\Omega I_t \, dx\right] dt = \int_0^T \theta(t) \left[\int_\Omega I_t \operatorname{div} b_t \, dx\right] dt + \int_0^T \theta(t) \left[\int \tau_t \, d\operatorname{Div}^j b_t\right] dt \\ + \theta(0) \int_\Omega \tau_0 \, dx - \theta(T) \int_\Omega \tau_T \, dx. \end{aligned} \tag{108}$$

In particular, $t \mapsto \int I_t \, dx$ is absolutely continuous on $(0, T)$.

Proof. Choose $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi = 1$ in $K \Supset \Omega$. Set $\varphi(x, t) := \theta(t)\psi(x)$. Writing out $F(I, u; \tau) = 0$, we have

$$-\int_{\Omega^T} \langle \nabla\varphi, Iu\rangle \, d(t, x) - \int_{\Omega^T} \varphi I \operatorname{div} u \, d(t, x) - \int \varphi\tau \, d\operatorname{Div}^j u = 0.$$

Because $\nabla\varphi(x, t) = (\theta'(t)\psi(x), \nabla\psi(x)\theta(t))$, we obtain

$$\begin{aligned} -\int_0^T \theta'(t) \int_\Omega \psi I_t \, dx \, dt - \int_0^T \theta(t) \int_\Omega \langle \nabla\psi, I_t b_t\rangle \, dx \, dt \\ - \int_0^T \theta(t) \int_\Omega \psi I_t \operatorname{div} b_t \, dx \, dt - \int \theta(t)\psi(x)\tau(t, x) \, d\operatorname{Div}^j u(t, x) = 0. \end{aligned}$$

Employing the fact that $\psi = 1$ on $K \Supset \Omega$, this reduces into

$$-\int_0^T \theta'(t) \int_\Omega I_t \, dx \, dt - \int_0^T \theta(t) \int_\Omega I_t \operatorname{div} b_t \, dx \, dt - \int \theta(t)\tau(t, x) \, d\operatorname{Div}^j u(t, x) = 0.$$

Thus (108) follows if

$$\int \theta(t)\tau(t, x) \, d\operatorname{Div}^j u(t, x) = \int_0^T \theta(t) \left[\int \tau_t \, d\operatorname{Div}^j b_t\right] dt + \theta(0) \int_\Omega \tau_0 \, dx - \theta(T) \int_\Omega \tau_T \, dx. \tag{109}$$

To show (109), we will employ the Structure Theorem. Towards this end, we let $(\xi_0, \ldots, \xi_n)$ be the standard basis of $\mathbb{R}^{n+1}$. Then $\operatorname{Div}^j u = \sum_{i=0}^n \langle E^j u\xi_i, \xi_i\rangle$, where, according to Theorem 1, for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, it holds

$$\langle E^j u\xi, \xi\rangle(\varphi) = \int_{\xi^\perp} \left(\int \varphi(y + t\xi) \, dD^j u^{[y,\xi]}(t)\right) d\mathcal{H}^n(y). \tag{110}$$

Additionally, for $\mathcal{H}^n$-a.e. $y \in \xi^\perp$, we have

$$J_{u^{[y,\xi]}} = J_{u,\xi}^{[y,\xi]} = \{t \in \mathbb{R} \mid x = y + t\xi \in J_u, \langle u^+(x) - u^-(x), \xi\rangle \neq 0\}$$

as well as $(u^{[y,\xi]})^\pm(t) = \langle u^\pm(y+t\xi), \xi\rangle$ for all $t \in J_{u,\xi}^{[y,\xi]}$. The normals are oriented so that $\langle \nu_{J_u}, \xi\rangle \geq 0$ if and only if $\nu_{J_{u^{[y,\xi]}}} = 1$. In particular, we may observe for $\mathcal{H}^n$-a.e. $y \in \xi^\perp$ that

$$\begin{aligned} D^j u^{[y,\xi]} &= \langle (u^{[y,\xi]})^+ - (u^{[y,\xi]})^-, \nu_{J_{u^{[y,\xi]}}}\rangle \mathcal{H}^0 \llcorner J_{u^{[y,\xi]}} \\ &= \langle u^+(y + t\xi) - u^-(y + t\xi), \xi\rangle \nu_{J_{u^{[y,\xi]}}} \mathcal{H}^0 \llcorner J_{u,\xi}^{[y,\xi]}. \end{aligned} \tag{111}$$

We now let $\xi = \xi_0 = (1, 0, \ldots, 0)$. Then $\xi^\perp = \{0\} \times \mathbb{R}^n$. Because $\langle u, \xi_0 \rangle = 1$ on Ω^T, (111) vanishes except at $t = 0$ or $t = T$ for $y \in \{0\} \times \Omega$. Moreover, $\langle u, \xi_0 \rangle = 0$ a.e. on $\mathbb{R}^{n+1} \setminus \Omega^T$. We therefore have for $y \in \{0\} \times \Omega$ that

$$D^j u^{[y,\xi_0]} = \langle u^+(y + 0\xi_0), \xi_0 \rangle \mathcal{H}^0 \llcorner \{0\} - \langle u^-(y + T\xi_0), \xi_0 \rangle \mathcal{H}^0 \llcorner \{T\} = \mathcal{H}^0 \llcorner \{0\} - \mathcal{H}^0 \llcorner \{T\},$$

while $D^j u^{[y,\xi_0]} = 0$ for $\mathcal{H}^n$-a.e. $y \in \xi^\perp \setminus (\{0\} \times \Omega)$. Here we have oriented ν_{J_u} on $\{0, T\} \times \Omega$ to equal ξ_0. Consequently, by application of (110), for $\varphi \in C_c(\mathbb{R}^{n+1})$ and $\varphi_s := \varphi(s, \cdot)$, we obtain

$$\langle E^j u \xi_0, \xi_0 \rangle(\varphi) = \int_\Omega \varphi_0(\hat{y})\, d\hat{y} - \int_\Omega \varphi_T(\hat{y})\, d\hat{y}.$$

On the other hand, when $\xi = (0, \zeta) \in \{\xi_1, \ldots, \xi_n\}$, and $y = (s, \hat{y})$, we may write

$$u(y + t\xi) = (1, b(s, \hat{y} + t\zeta)) \quad \text{and} \quad u^{[y,\xi]} = (b_s)^{[\hat{y},\zeta]}.$$

Now note that it follows from [2, Proposition 3.4] that $b_s \in \mathrm{BD}(\mathbb{R}^n)$ for $\mathcal{H}^1$-a.e. $s \in [0, T]$. Observe also that $\xi^\perp = \mathbb{R} \times \zeta^\perp$. Therefore, applying (110) and Fubini's theorem on u and b_s, we find for $\varphi \in C_c(\mathbb{R}^{n+1})$ that

$$\begin{aligned}
\langle E^j u \xi, \xi \rangle(\varphi) &= \int_{\xi^\perp} \left(\int \varphi(y + t\xi)\, dD^j u^{[y,\xi]}(t) \right) d\mathcal{H}^n(y) \\
&= \int \int_{\zeta^\perp} \left(\int \varphi(s, \hat{y} + t\zeta)\, dD^j (b_s)^{[\hat{y},\zeta]}(t) \right) d\mathcal{H}^{n-1}(\hat{y})\, ds \\
&= \int \langle E^j b_s \zeta, \zeta \rangle(\varphi_s)\, ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{Div}^j u(\varphi) &= \sum_{i=0}^n \langle E^j u \xi_i, \xi_i \rangle(\varphi) \\
&= \sum_{i=1}^n \int \langle E^j b_s \zeta_i, \zeta_i \rangle(\varphi_s)\, ds + \langle E^j u \xi_0, \xi_0 \rangle(\varphi) \\
&= \int \operatorname{Div}^j b_s(\varphi_s)\, ds + \int_\Omega \varphi_0(\hat{y})\, d\hat{y} - \int_\Omega \varphi_T(\hat{y})\, d\hat{y}.
\end{aligned} \tag{112}$$

This implies (109), completing the proof. □

Remark 6. From (112) one may observe that the divergence of u is, in a sense, "absolutely continuous in time" in $(0, T)$. The discontinuities at 0 and F correspond to the initial condition and the "final result", which are both subsumed into the "jump variable" τ.

We now have the following Gronwall estimate.

Lemma 5. *Let $I \in X_I$, $u = (1, b) \in X_u$, and $\tau \in L^1(\operatorname{Div}^j u)$ with $F(I, u; \tau) = 0$. Suppose $\int_0^T \|\max\{0, \operatorname{div} b_t\}\|_{L^\infty(\Omega)}\, dt < \infty$, and $I \geq 0$. Defining $\eta(t) := \int I_t(x)\, dx$, we then have*

$$\eta(t) \leq e^{\int_0^t \|\max\{0, \operatorname{div} b_s\}\|_{L^\infty(\Omega)}\, ds} \Big[\int_\Omega \tau_0\, dx + \int_0^t \int \tau_s\, d\operatorname{Div}^j b_s\, ds \Big], \quad t \in [0, T]. \tag{113}$$

Proof. It follows from the absolute continuity assertion and (108) of Lemma 4 that

$$\eta'(t) = \int_\Omega I_t \operatorname{div} b_t \, dx + \int \tau_t \, d \operatorname{Div}^j b_t \quad \text{for a.e. } t \in (0, T).$$

Employing the assumption $I \geq 0$, we deduce

$$\eta'(t) \leq \| \max\{0, \operatorname{div} b_t\} \|_{L^\infty(\Omega)} \eta(t) + \int \tau_t \, d \operatorname{Div}^j b_t. \quad \text{for a.e. } t \in (0, T).$$

By application of Gronwall's lemma

$$\eta(t) \leq e^{\int_0^t \| \max\{0, \operatorname{div} b_s\} \|_{L^\infty(\Omega)} \, ds} \big[\eta(0) + \int_0^t \int \tau_s \, d \operatorname{Div}^j b_s \, ds\big] \quad \text{for } t \in [0, T].$$

Since η is zero outside $[0, T]$, inspecting the jumps on the right hand side of (108) shows that the distributional trace of η at 0 is $\int_\Omega \tau_0 \, dx$. Thus (113) follows. □

Proposition 3. *Suppose $I \in X_I$, $u \in X_u$, and $\tau \in L^1(\operatorname{Div}^j u)$ with $F(I, u; \tau) = 0$ and $I \geq 0$. Suppose also $\int_0^T \| \max\{0, \operatorname{div} b_t\} \|_{L^\infty(\Omega)} \, dt < \infty$. Then $I = 0$ (a.e.), if $\tau = 0$ (a.e.) on $L_u^+ \cup L_u^-$.*

Proof. The claim follows by direct application of Lemma 5, if we show

$$\int_\Omega \tau_0 \, dx + \int_0^t \int \tau_s \, d \operatorname{Div}^j b_s \, ds \leq 0.$$

Minding (109), this amounts to showing $\tau \operatorname{Div}^j u \leq 0$. We indeed have $\tau \operatorname{Div}^j u \llcorner (L_u^+ \cup L_u^-) \leq 0$ by the assumption $\tau = 0$ on $L_u^+ \cup L_u^-$, so it remains to show $\tau \operatorname{Div}^j u \llcorner (J_u^{\operatorname{div}} \setminus (L_u^+ \cup L_u^-)) \leq 0$.

From Proposition 3 we deduce that $I^\pm$ exists a.e. on $N_u^\pm \cup P_u^\pm$ (which we recall being defined in (51),(52)), and

$$\tau \langle u^+ - u^-, \nu_{J_u} \rangle = \langle I^+ u^+ - I^- u^-, \nu_{J_u} \rangle \quad \mathcal{H}^n\text{-a.e. on } J_u, \tag{114}$$

with $I^\pm \geq 0$ defined arbitrarily on $J_u \setminus (N_u^\pm \cup P_u^\pm)$. Now, on $J_u^{\operatorname{div}} \setminus (L_u^+ \cup L_u^-) = J_u^{\operatorname{div}} \setminus (P_u^+ \cup P_u^-)$, we have both $\langle u^+, \nu_{J_u} \rangle \leq 0$ and $\langle u^-, -\nu_{J_u} \rangle \leq 0$. Therefore, $I \geq 0$ and (114) imply $\tau \langle u^+ - u^-, \nu_{J_u} \rangle \leq 0$ a.e. on $J_u^{\operatorname{div}} \setminus (L_u^+ \cup L_u^-)$. This means $\tau \operatorname{Div}^j u \llcorner (J_u^{\operatorname{div}} \setminus (L_u^+ \cup L_u^-)) \leq 0$. We may thus conclude that $\tau \operatorname{Div}^j u \leq 0$, as required. □

With the help of the renormalisation idea due to DiPerna and Lions [3], we can forgo the assumption $I \geq 0$, and thus show uniqueness with respect to boundary conditions and jumps. The definition of κ_u may be recalled from (91). Observe that $\kappa_u \in L^1(\operatorname{Div}^j u)$ holds automatically when $\mathcal{H}^n(J_u) < \infty$, because u is bounded.

Lemma 6. *Let $u \in X_u$ and $I \in X_I$, and suppose $\kappa_u \in L^1(\operatorname{Div}^j u)$ as well as $0 \in F(I, u; \tau)$ for some $\tau \in L^1(\operatorname{Div}^j u)$. Then $F(\beta(I), u; \tau_\beta) = 0$ for some $\tau_\beta \in L^1(\operatorname{Div}^j u)$ for all Lipschitz functions $\beta \in C^1(\mathbb{R})$.*

Proof. The proof is a rather straightforward application of the chain rule [14, 13] for divergences of composition of the form $\beta(I)u$. First of all, we observe from the condition $0 \in F(I, u)$ that

$\mathrm{Div}(Iu)$ is a measure with finite variation; cf. Proposition 3. Accordingly, by [14] the absolutely continuous part of the distributional divergence $\mathrm{Div}(\beta(I)u)$ can be written

$$\operatorname{div}(\beta(I)u) = \big(\beta(I) - I\beta'(I)\big)\operatorname{div} u + \beta'(I)\operatorname{div}(Iu), \tag{115}$$

while the singular part satisfies for any oriented countably $\mathcal{H}^n$-rectifiable Σ with normal field ν the condition

$$\mathrm{Div}(\beta(I)u)\llcorner\Sigma = \left[\mathrm{Tr}^+(u,\Sigma)\beta\left(\frac{\mathrm{Tr}^+(Iu,\Sigma)}{\mathrm{Tr}^+(u,\Sigma)}\right) - \mathrm{Tr}^-(u,\Sigma)\beta\left(\frac{\mathrm{Tr}^-(Iu,\Sigma)}{\mathrm{Tr}^-(u,\Sigma)}\right)\right]\mathcal{H}^n\llcorner\Sigma. \tag{116}$$

When $\mathrm{Tr}^\pm(u,\Sigma) = 0$, the corresponding argument of β is defined arbitrarily here. Moreover, if $\mathrm{Div}^j(Iu)$ is concentrated on a countably $\mathcal{H}^n$-rectifiable set Σ, then $\mathrm{Div}^j(\beta(I)u)$ is concentrated on Σ.

Now, regarding the absolutely continuous part, since $0 \in F(I,u)$, we have $\operatorname{div}(Iu) = I\operatorname{div} u$. Therefore also $\operatorname{div}(\beta(I)u) = \beta(I)\operatorname{div}$ by (115). Thus the absolutely continuous part of the condition $0 \in F(\beta(I),u)$ has been taken care of.

As for the jump part, from above we have $\mathrm{Div}^j(\beta(I)u) \ll \mathrm{Div}^j(Iu)$, while $0 \in F(I,u)$ implies $\mathrm{Div}^j(Iu) \ll \mathrm{Div}^j u$. It follows that $\mathrm{Div}^j(\beta(I)u) = \tau_\beta \mathrm{Div}^j u$, for some measurable function τ_β defined on J_u^{div}. We have to show that $\tau_\beta \in L^1(\mathrm{Div}^j u)$. Minding Proposition 3, the one-sided Lebesgue limits $I^\pm$ exist a.e. when $\langle u^\pm, \nu_{J_u}\rangle \neq 0$, and $(Iu)^\pm = I^\pm u^\pm$. Therefore we may simplify (116) to

$$\mathrm{Div}(\beta(I)u)\llcorner\Sigma = \big[\langle u^+, \nu_{J_u}\rangle\beta(I^+) - \langle u^-, \nu_{J_u}\rangle\beta(I^-)\big]\,\mathcal{H}^n\llcorner(J_u^{\mathrm{div}} \cap \Sigma). \tag{117}$$

Observe now that, a.e. on J_u^{div}, we have

$$|\tau_\beta\langle u^+ - u^-, \nu_{J_u}\rangle| = |\langle u^+, \nu_{J_u}\rangle\beta(I^+) - \langle u^-, \nu_{J_u}\rangle\beta(I^-)| \le M\kappa_u|\langle u^+ - u^-, \nu_{J_u}\rangle|,$$

where $M := \max\beta([-M_I, M_I]) < \infty$. When $\kappa_u \in L^1(\mathrm{Div}^j u)$, as we have assumed, it thus follows that $\tau_\beta \in L^1(\mathrm{Div}^j u)$.

Finally, it remains to show that $\mathrm{Div}^c(\beta(I)u)$ vanishes. This is not directly covered by the results of [14], but can be obtained as follows. First of all, denoting $E^s := E^j + E^c$, and $\mathrm{Div}^s := \mathrm{Div}^j + \mathrm{Div}^c$, by the proof of [14, Theorem 3.3], $\mathrm{Div}^s(\beta(I)u)$ is the limit, in the sense of distributions, of

$$C_1^\delta + C_2^\delta + C_3^\delta := \beta'(I_\delta)(\mathrm{Div}^s(Iu) * \rho_\delta) + [\beta(I_\delta) - I_\delta\beta'(I_\delta)]\,\mathrm{Div}^s u + \beta'(I_\delta)T_\rho \quad \text{as } \delta \searrow 0. \tag{118}$$

Here $\rho_\delta := \delta^{n+1}\rho(\cdot/\rho)$ are standard the mollifiers on $\mathbb{R}^{n+1}$, the commutator

$$T_\delta := \mathrm{Div}(Iu) * \rho_\delta - \mathrm{Div}(I(u * \rho_\delta)),$$

and $I_\delta := I * \rho_\delta$. By [14, Proposition 3.4], any weak* limit σ of $\{|T_\delta|\}$ is a singular measure satisfying $\sigma\llcorner A \le \|I\|_{L^\infty(A)}L|E^s u|$ for any Borel set A and a constant L dependent on ρ and n. Since $u \in \mathrm{SBD}(\mathbb{R}^{n+1})$, and $I \in L^\infty(\mathbb{R}^{n+1})$, we get $\sigma \ll |E^j u|$. In particular, any limit of C_3^δ as $\delta \searrow 0$ is absolutely continuous with respect to $|E^j u|$. We also have that any limit of C_1^δ as $\delta \searrow 0$ is absolutely continuous with respect to $\mathrm{Div}^s(Iu) = \mathrm{Div}^j(Iu) \ll \mathrm{Div}^j u$, and any limit of C_2^δ is absolutely continuous with respect to $\mathrm{Div}^s u = \mathrm{Div}^j u$. It thus follows that $\mathrm{Div}^s(\beta(I)u) \ll |E^j u|$. But $E^j u$ is concentrated on the countably $\mathcal{H}^n$-rectifiable set J_u, and the Cantor part of $E(\beta(I)u)$ vanishes on such sets. Hence $\mathrm{Div}^c(\beta(I)u)$ vanishes, so $\mathrm{Div}^s(\beta(I)u) = \mathrm{Div}^j(\beta(I)u)$. The claim follows. □

We have finally reached our main uniqueness result.

Theorem 6. *Suppose $u \in X_u$ with $\int_0^T \|\max\{0, \operatorname{div} b_t\}\|_{L^\infty(\Omega)}\,dt < \infty$ and $\kappa_u \in L^1(\operatorname{Div}^j u)$. Then, given $Y^\pm \in L^\infty_{M_I}(L_u^\pm)$, there is at most one solution pair $I \in X_I$ and $\tau \in L^1(\operatorname{Div}^j u)$ of $0 \in F(I, u; \tau)$ with one-sided traces satisfying $I^+ = Y^+$ on L_u^+ and $I^- = Y^-$ on L_u^-.*

Proof. Observe, first of all, that by Proposition 3 the one-sided Lebesgue limits $I^\pm$ of I exist a.e. on $L_u^\pm$. Suppose then that there are two solutions $I, I' \in X_I$ and $\tau, \tau' \in L^1(\operatorname{Div}^j u)$ satisfying $T(I, u; \tau) = 0$ and $T(I', u; \tau') = 0$ with $I^\pm = Y^\pm$ on $L_u^\pm$ and with $(I')^\pm = Y^\pm$ on $L_u^\pm$. In particular, $T(I - I', u; \tau - \tau') = 0$ with $(I - I')^\pm = (Y^\pm - Y^\pm) = 0$ on $L_u^\pm$.

Now, according to Lemma 6, $I - I'$ is a renormalised solution, i.e., given, e.g., $\beta(t) := |t|^2/(1 + |t|)$, we have $0 \in F(\beta(I - I'), u; \tau_\beta)$ for some $\tau_\beta \in L^1(\operatorname{Div}^j u)$. Recalling that

$$L_u^\pm = \{x \in J_u^{\mathrm{div}} \mid \langle u^\pm(x), \pm\nu(x)\rangle > 0\},$$

and observing that $\beta \geq 0$, an inspection of (117) now reveals that

$$\operatorname{Div}(\beta(I - I')u)\llcorner J_u^{\mathrm{div}} \leq 0.$$

But thanks to $F(\beta(I - I'), u; \tau_\beta) = 0$, we have $\operatorname{Div}(\beta(I - I')u)\llcorner J_u^{\mathrm{div}} = \tau_\beta \operatorname{Div}^j u$, so it follows that $\tau_\beta \operatorname{Div}^j u \leq 0$. A direct application of Lemma 5, similarly to Proposition 3, therefore shows that $\beta(I' - I) = 0$ (a.e.). Thus $I = I'$ (a.e.). Moreover, τ is easily seen to be uniquely determined (a.e.) by I and u on J_u^{div}. The solution I, τ must therefore be unique. $\square$

Remark 7. Because $I - I'$ may be negative, it is not sufficient to assume that $\tau - \tau' = 0$ on $L_u^+ \cup L_u^-$, as in Proposition 3. Just consider $u(t, x) = (1, \operatorname{sgn} x)$ in $\Omega^T := (0, T) \times (-1, 1)$. Then $J_u = [0, T] \times \{0\} \cup \partial\Omega^T$, and $L_u^+ \cup L_u^- = [0, T] \times \{0\} \cup \{0\} \times [-1, 1]$. Moreover, $L_u^+ \cap L_u^- = [0, T] \times \{0\}$. Given any $\alpha \in \mathbb{R}$, let us set $I_\alpha(t, x) := \pm\alpha$ for $\pm x \leq t \leq T$, and $I_\alpha(t, x) := 0$ elsewhere in $[0, T] \times [-1, 1]$. Then I_α is a solution of $0 \in F(I, u)$ with $\tau = 0$ on $L_u^+ \cup L_u^-$.

In the case of "at most one-sided sources" with not both $\langle u^+, \nu_{J_u}\rangle > 0$ and $-\langle u^-, \nu_{J_u}\rangle > 0$, it is easy to see *formally* that it suffices to assume $\tau = \tau'$ on $L_u^+ \cup L_u^-$. To see this, note that $F(I - I', u; \tau - \tau') = 0$ then implies

$$(I - I')^+ \langle u^+, \nu_{J_u}\rangle - (I - I')^- \langle u^-, \nu_{J_u}\rangle = 0 \quad \text{on } L_u^+ \cup L_u^-.$$

Thus, when $\langle u^\mp, \nu_{J_u}\rangle = 0$, trivially $(I - I')^\pm = 0$ on $L_u^+ \cup L_u^-$. Otherwise, when both $\langle u^+, \nu_{J_u}\rangle \neq 0$ and $\langle u^-, \nu_{J_u}\rangle \neq 0$, we deduce $\operatorname{sgn}(I - I')^+ = \operatorname{sgn}(I - I')^-$. Consequently, with $\beta(t) = |t|$ (which is not admissible for Lemma 6), we get

$$\beta((I - I')^+)\langle u^+, \nu_{J_u}\rangle - \beta((I - I')^-)\langle u^-, \nu_{J_u}\rangle = \pm[(I - I')^+ \langle u^+, \nu_{J_u}\rangle - (I - I')^- \langle u^-, \nu_{J_u}\rangle] = 0$$

on $L_u^+ \cup L_u^-$. An inspection of (117) would now, *formally*, show that $\operatorname{Div}(\beta(I - I')u)\llcorner J_u \leq 0$. An approximation argument on β could be used to establish this more rigorously.

4. The image interpolation problem

4.1. Problem formulation

We now intend to study the problem (3) of fitting to available data a space-time image I satisfying our generalisation $0 \in T(I, u)$ of the optical flow constraint for some SBD velocity field

u. Such an "optimal control" approach to the optical flow problem has been previously studied in [15] in a Sobolev space setting.

Let $\alpha, \beta \geq 0$ and $\theta, \gamma > 0$. Suppose $\psi : [0, \infty) \to [0, \infty)$ is convex, increasing, and satisfies $\psi(t)/t \to \infty$ as $t \to \infty$. Suppose $\Psi_d : \Omega^T \times \mathbb{R} \to [0, \infty)$ is Borel measurable, and that $\Psi_d(x, \cdot)$ is convex and continuous for a.e. $x \in \Omega^T$. With $\eta : \mathcal{M}(\Omega) \to \mathbb{R}$ yet to be determined, we then consider the functional

$$\begin{aligned} J(I, u) := \int_{\Omega^T} \Psi_d(y, I(y))\, d\mathcal{L}^{n+1}(y) + \theta |DI|(\Omega^T) \\ + \alpha \|u\|_{L^1} + \beta |E^j u|(\mathbb{R}^{n+1}) + \int \psi(|\mathcal{E}u|)\, d\mathcal{L}^{n+1} + \eta(\operatorname{Div}^j u) + \gamma \mathcal{H}^n(J_u), \end{aligned} \tag{J}$$

and the problem

$$\min J(I, u) \quad \text{subject to} \quad I \in X_I,\ u \in X_u, \text{ and } 0 \in F(I, u). \tag{P}$$

The first term in (J), involving Ψ_d, is the data-fitting term, and the rest are regularisation terms.

Example 2. Typically Ψ_d is taken to measure the distance to available data. For example,

$$\Psi_d(x, s) = \begin{cases} \|I_d(x) - s\|^2/2, & x \in \Omega_d, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Omega_d \subset \Omega^T$ is an open set where data is available, and I_d is the data. As a particular case, when data is available at times $t_1 \leq t_2 \leq \ldots \leq t_n \in [0, T]$ with measurement accuracy (voxel length in time) δ, we might have $\Omega_d = \bigcup_{i=1}^n (t_i, t_i + \delta) \times \Omega$.

4.2. Divergence regularisation

We would like to show the existence of solutions to (P). Towards this end, we need to ensure that any minimising sequence $\{(I^i, u^i)\}_{i=0}^\infty$ admits a subsequence converging in the sense required by Theorem 4, showing the outer-semicontinuity of F. This will be guaranteed by the regularisation terms of (J), if we define η appropriately. More precisely, we need some way to force (40), that is, $|\operatorname{Div}^j u^i|(\mathbb{R}^{n+1}) \to |\operatorname{Div}^j u|(\mathbb{R}^{n+1})$.

One simple approach would be to require that for a given $\delta > 0$, we would have $|\operatorname{Div}^j(B(y, \delta))| = |\operatorname{Div}^j|(B(y, \delta))$ for all $y \in \mathbb{R}^{n+1}$. That is, in each ball of radius δ, the density of $\operatorname{Div}^j u$ with respect to $\mathcal{H}^n$ would either be a.e. negative or a.e. positive. This would keep the positive and negative parts of the measure apart and prevent cancellation at the limit. However, we do not need to force such strong separation, and can instead penalise based on the same idea. This is how we will construct in the next proposition the yet undetermined term $\eta(\operatorname{Div}^j u)$ of (J).

Definition. A sequence $\{(f_j, \nu_j)\}_{j=0}^\infty$ of bounded Borel functions $f^j : \mathbb{R}^m \to \mathbb{R}$ with compact support and continuous in $\mathbb{R}^m \setminus S_f$, along with Borel probability measures ν^j on $\mathbb{R}^m$ is said to form a *nested sequence of functions* if $f_j(x) = \int f_{j+1}(x - y)\, d\nu_j(y)$ (a.e.).

Proposition 4. *Let $\Omega \subset \mathbb{R}^m$ be an open bounded set, and $\{(f_j, \nu_j)\}_{j=0}^\infty$ a nested sequence of functions such that $f_j \geq 0$, and $\int f_j\, dx = 1$. For $\mu \in \mathcal{M}(\mathbb{R}^m)$, set*

$$\eta(\mu) := \sum_{j=0}^\infty \int_{\mathbb{R}^m} |\mu|(\tau_x f_j) - |\mu(\tau_x f_j)|\, dx, \qquad \textit{where} \qquad \tau_x f(y) := f(y - x).$$

Suppose $\{\mu^i\}_{i=0}^\infty \subset \mathcal{M}(\mathbb{R}^m)$ *weakly* converges to* $\mu \in \mathcal{M}(\mathbb{R}^m)$ *with* $\operatorname{supp}\mu^i \subset \Omega$ *and* $\sup_i |\mu^i|(\Omega) + \eta(\mu^i) < \infty$*. If also* $|\mu^i| \overset{*}{\rightharpoonup} \lambda$*, then* $\lambda = |\mu|$*. Moreover, the functional* η *is lower-semicontinuous with respect to the simultaneous weak* convergence of* $\{(\mu^i, |\mu^i|)\}_{i=0}^\infty$*.*

If each $f_j \in C_c(\mathbb{R}^m)$*, then it is not necessary to assume the weak* convergence of* $|\mu^i|$ *to* λ*.*

Proof. Observe that by application of Fubini's theorem and the assumption $\int f_j\,dx = 1$, we have

$$\int_{\mathbb{R}^m} |\mu|(\tau_x f_j)\,dx = \int_\Omega \int_{\mathbb{R}^m} f_j(y-x)\,dx\,d|\mu|(y) = |\mu|(\Omega).$$

Hence, we may alternatively write

$$\eta(\mu) = \sum_{j=0}^\infty \eta_j(\mu), \quad \text{where} \quad \eta_j(\mu) := |\mu|(\Omega) - \int |\mu(\tau_x f_j)|\,dx. \tag{119}$$

Recall that S_f denotes the set of (approximate) discontinuity points of f. Fubini's theorem and the fact that S_f is an $\mathcal{L}^m$-negligible Borel set, imply that $\int \lambda(S_{\tau_x f_j})\,dx = 0$. This shows that $\lambda(S_{\tau_x f_j}) = 0$ for a.e. $x \in \mathbb{R}^m$. As a consequence (see, e.g., [2, Proposition 1.62]), we have $\mu^i(\tau_x f_j) \to \mu(\tau_x f_j)$ for a.e. $x \in \mathbb{R}^m$. Minding that $\sup_i |\mu^i|(\Omega) < \infty$ and Ω is bounded by assumption, an application of the dominated convergence theorem then shows that

$$\int |\mu^i(\tau_x f_j)|\,dx \to \int |\mu(\tau_x f_j)|\,dx, \quad (i \to \infty). \tag{120}$$

We stress that (120) holds because of the convergence $|\mu^i| \overset{*}{\rightharpoonup} \lambda$. Since the total variation $|\mu|(\Omega)$ is lower-semicontinuous with respect to weak* convergence, it follows from (120) that each η_j is lower-semicontinuous with respect to the simultaneous weak* convergence of $\{(\mu^i, |\mu^i|)\}_{i=0}^\infty$. Consequently also η is lower-semicontinuous.

If f_j is actually continuous with compact support, then $\mu^i(\tau_x f_j) \to \mu(\tau_x f_j)$ for all $x \in \Omega$ by the weak* convergence of μ^i to μ alone, so (120) and lower-semicontinuity holds without assumptions on the convergence of $\{|\mu^i|\}_{i=0}^\infty$.

Observe now that thanks to the fact that $\{(f_j, \nu_j)\}_{i=0}^\infty$ is a nested sequence of functions, $\{\eta_j(\mu)\}_{j=0}^\infty$ forms a decreasing sequence (for any $\mu \in \mathcal{M}(\Omega)$). Indeed, as $f_j(x) = \int f_{j+1}(x-y)\,d\nu_j(y)$ and $\nu_j(\mathbb{R}^m) = 1$ with $\nu_j \geq 0$, we have

$$\begin{aligned}\int |\mu(\tau_x f_j)|\,dx &= \int \left| \int \mu(\tau_{x+y} f_{j+1})\,d\nu_j(y) \right| dx \leq \int \int |\mu(\tau_{x+y} f_{j+1})|\,d\nu_j(y)\,dx \\ &= \int \int |\mu(\tau_{x+y} f_{j+1})|\,dx\,d\nu_j(y) = \int |\mu(\tau_x f_{j+1})|\,dx\end{aligned}$$

after a change of variables in the last step to eliminate y. Minding the definition (119), it follows from here that $\eta_j(\mu) \geq \eta_{j+1}(\mu)$.

To show the convergence of the total variation measures $|\mu^i|$ to $|\mu|$, we only have to show $|\mu^i|(\Omega) \to |\mu|(\Omega)$. To see this, we choose an arbitrary $\epsilon > 0$, and write

$$|\mu|(\Omega) - |\mu^i|(\Omega) = \eta_j(\mu) - \eta_j(\mu^i) + \int |\mu(\tau_x f_j)| - |\mu^i(\tau_x f_j)|\,dx. \tag{121}$$

Next we observe from the already proved lower semi-continuity of η and the bound $\sup_i \eta(\mu^i) =: K < \infty$ that $\eta(\mu) \leq K$ as well. Therefore, recalling that $\{\eta_j(\mu)\}_{j=1}^\infty$ and $\{\eta_j(\mu^i)\}_{j=1}^\infty$ for $i =$

$0, 1, 2, \ldots$ are decreasing sequences, as shown above, it follows that by taking j large enough, we can ascertain that $\sup\{\eta_j(\mu), \eta_j(\mu^1), \eta_j(\mu^2), \ldots\} \leq \epsilon$. (Note that $\eta_j \geq 0$!) Employing this observation in (121), we find that

$$\big||\mu|(\Omega) - |\mu^i|(\Omega)\big| \leq 2\epsilon + \left|\int |\mu(\tau_x f_j)| - |\mu^i(\tau_x f_j)|\, dx\right|$$

for any large enough j and all i. The integral term tends to zero as $i \to \infty$ by (120). Therefore, we have

$$\lim_{i\to\infty} ||\mu^i|(\Omega) - |\mu|(\Omega)| \leq 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof can be concluded. □

Remark 8. Let the functions $f_j \geq 0$ be in $C_c(\mathbb{R}^m)$ and instead of $\int f_j\, dx = 1$, satisfy $\sum_{\xi\in\delta_j\mathbb{Z}^m} \tau_\xi f_j \equiv 1$ for some $\delta_j > 0$. Proposition 4 then holds with nearly identical proof if we define

$$\eta(\mu) := \sum_{j=0}^{\infty} \sum_{\xi\in\delta_j\mathbb{Z}^m} (|\mu|(\tau_\xi f_j) - |\mu(\tau_\xi f_j)|) = \sum_{j=0}^{\infty}\left(|\mu|(\Omega) - \sum_{\xi\in\delta_j\mathbb{Z}^m} |\mu(\tau_\xi f_j)|\right).$$

Example 3. The following form nested sequences of functions satisfying the conditions $f_j \geq 0$ and $\int f_j\, dx = 1$.

1. The indicator functions $f_j := 2^{jm}\chi_{2^{-j}Q}$, where $Q := [0, 1]^m$.
2. On $\mathbb{R}$, the triangular functions $f_j(x) := 2^j f(2^j x)$, where $f(x) = \max\{0, 1 - |x|\}$. On $\mathbb{R}^m$ we can similarly take a more complicated (shape) function related regular simplicial meshes, and appropriate weights for f_j.
3. For a decreasing sequence $\delta_j \searrow 0$, the mollifiers $f_j := \zeta_{\delta_j}$, when the semigroup property $\zeta_{\epsilon+\delta} = \zeta_\epsilon * \zeta_\delta$ is satisfied.

Example 4. Let us take $f_j := 2^{2j}\chi_{2^{-j}Q}$, where $Q := [0, 1]^2$, as above. We also let $R := \{0\}\times[0, 1] \subset \mathbb{R}^2$, and $e := (1, 0)$. Then we study boundedness of $\eta(\mu^i)$ the following cases:

1. $\mu^i := \mathcal{H}^1\llcorner R - \mathcal{H}^1\llcorner(e/i + R)$. Now $|\mu^i|(\mathbb{R}^2) = 2$, but $\mu^i \overset{*}{\rightharpoonup} 0$, so by Proposition 4 necessarily $\eta(\mu^i) \to \infty$.
2. $\mu^i := \mathcal{H}^1\llcorner R - (1/i)\mathcal{H}^1\llcorner(e/i + R)$. This time $|\mu^i|(\mathbb{R}^2) = 1 + 1/i$, and $\mu^i \overset{*}{\rightharpoonup} \mathcal{H}^1\llcorner R$, so it would be desirable to have $\sup_i \eta(\mu^i) < \infty$. Let us verify that this is indeed the case. For each x such that the square $x + 2^{-j}Q$ touches both R and $e/i + R$, we have

$$\begin{aligned}|\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)| &= |2^{2j}\mu^i|(x + 2^{-j}Q) - |2^{2j}\mu^i(x + 2^{-j}Q)| \\ &\leq 2^{2j}\big((1 + 1/i)2^{-j} - |(1 - 1/i)2^{-j}|\big) \leq 2^{j+1}/i.\end{aligned}$$

   Such $x = (x_1, x_2)$ must satisfy $1/i - 2^{-j} \leq x_1 \leq 0$ and $-2^{-j} \leq x_2 \leq 1$. As squares that do not touch both R and $e/i + R$ do not contribute to η_j, this gives

$$\int |\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)|\, dx \leq \max\{0, 2^{-j} - 1/i\}(1 + 2^{-j})2^{j+1}/i \leq (6/i)\max\{0, 1 - 2^j/i\}.$$

   Since this is non-zero only for $j < \log_2 i$, summing over j, we have $\eta(\mu^i) \leq 6(\log_2 i + 1)/i$. Thus $\eta(\mu^i)$ is bounded for $i > 0$. In fact, it tends to zero as $i \to \infty$.

3. $\mu^i := \mathcal{H}^1 \llcorner R - \mathcal{H}^1 \llcorner (1/i)(e+R)$. Again $|\mu^i|(\mathbb{R}^2) = 1 + 1/i$, and $\mu^i \to \mathcal{H}^1 \llcorner R$, while for any x such that the square $x + 2^{-j}Q$ touches both R and $(1/i)(e+R)$ one has

$$|\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)| \leq 2^{2j} \cdot 2/i.$$

As such squares must satisfy $1/i - 2^{-j} \leq x_1 \leq 0$ and $-2^{-j} \leq x_2 \leq 2^{-j}$, it follows that

$$\int |\mu^i|(\tau_x f_j) - |\mu^i(\tau_x f_j)|\, dx \leq \max\{0, 2^{-j} - 1/i\} 2^{2j+1-j+1}/i = (4/i) \max\{0, 1 - 2^j/i\}.$$

Hence, as in Case 2, we get $\eta(\mu^i) \searrow 0$.

4.3. Existence of solutions

With η defined, we may finally conclude the paper with the following existence result.

Theorem 7. *Problem* (P) *admits a solution.*

Proof. Let $\{(I^i, u^i)\}_{i=0}^\infty$ be a minimising sequence for J. We may assume that $J(I^i, u^i) \leq K < \infty$. It follows that $\{(I^i, u^i)\}_{i=0}^\infty$ admits a subsequence, unrelabelled, such that $\{I^i\}_{i=0}^\infty$ is convergent weakly in $\mathrm{BV}(\Omega^T)$ to some $I \in X_I \cap \mathrm{BV}(\Omega^T)$. We now want to extract a further subsequence such that $\{u^i\}_{i=0}^\infty$ is also convergent in the senses (7)–(10) and (40).

We do this by applying Theorem 2 and Proposition 4. Even when $\alpha = 0$, we have an L^1 bound for u^i from $\mathcal{L}^{n+1}(\Omega^T) < \infty$ and $\|u^i\|_{L^\infty(\Omega^T)} \leq M_u$. Similarly we can bound $|E^j u|(\mathbb{R}^{n+1})$ when $\beta = 0$ by employing $\gamma \mathcal{H}^n(J_u) \leq K$ and $\gamma > 0$. Therefore, as $J(I, u)$ includes the remaining terms $\int \psi(\mathcal{E}u)\, dx$ and $\mathcal{H}^n(J_u)$ required to be bounded by Theorem 2, it follows that there is a further subsequence of $\{(I^i, u^i)\}_{i=0}^\infty$, unrelabelled, such that $\{u^i\}_{i=0}^\infty$ is convergent to some $u \in X_u$ in the sense (7)–(10). In particular, it follows from (9) that $\operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \operatorname{Div}^j u$ weakly* in $\mathcal{M}(\mathbb{R}^{n+1})$. By extracting a further subsequence, still unrelabelled, we may assume that $\{|\operatorname{Div}^j u^i|\}_{i=0}^\infty$ is weakly* convergent to some $\lambda \in \mathcal{M}(\mathbb{R}^{n+1})$. Observing the bound $\eta(u^i) \leq K$, Proposition 4 now shows that $\lambda = |\operatorname{Div}^j u|$. This proves (40).

The convergences (37)–(39) follow from (7)–(9). We have therefore shown that all the conditions of Corollary 1 hold, and so $0 \in F(I, u)$. It only remains to show that $J(I, u)$ is lower-semicontinuous with respect to weak convergence of $\{I^i\}_{i=0}^\infty$ in $\mathrm{BV}(\Omega^T)$ and the convergences (7)–(10),(40) of $\{u^i\}_{i=0}^\infty$. Most of this is standard. Since $\Psi_d(x, \cdot)$ is lower-semicontinuous for a.e. $x \in \Omega^T$, and Ψ_d is Borel measurable and bounded from below, $I \mapsto \int_{\Omega^T} \Psi_d(x, I(x))\, dx$ is lower-semicontinuous with respect to strong convergence in $L^1(\Omega^T)$; see, e.g., [21, Theorem 6.49]. It is well known that $|DI|(\Omega^T)$ is lower-semicontinuous with respect to weak convergence in $\mathrm{BV}(\Omega^T)$, while Proposition 4 provides the required lower-semicontinuity of η. Finally, the terms

$$\alpha \|u\|_{L^1} + \beta |E^j u|(\mathbb{R}^{n+1}) + \int \psi(|\mathcal{E}u|)\, d\mathcal{L}^{n+1} + \gamma \mathcal{H}^n(J_u)$$

related to Theorem 2 are lower-semicontinuous by, e.g., [8, Corollary 1.2]. This completes the proof. □

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Appendix A. Auxiliary results

Lemma 7. *Let $\mu \in \mathcal{M}(A)$, and suppose $v, v^0, v^1, \ldots \in L^1(\mu; \mathbb{R}^k)$ with $v^i \to v$ strongly. If $\mu(\{x \in A \mid v(x) = 0\}) = 0$, then $\lim_{i\to\infty} \mu(\{x \in A \mid v^i(x) = 0\}) = 0$.*

Proof. Let $\epsilon > 0$ be arbitrary. We assume the contrary of the claim: that for some $\delta > 0$ and each $i = 0, 1, 2, \ldots$, the sets $Z_i := \{x \in A \mid v^i(x) = 0\}$ satisfy $\mu(Z_i) \geq 2\delta$. Since L^1 convergence implies convergence in measure, we find that the sets $E^i := \{x \in A \mid \|v^i(x) - v(x)\| > \epsilon\}$ satisfy $\mu(E^j) < \delta$ for some large index j. Let $D_\epsilon := Z_j \setminus E^j$. We then have

$$\|v(x)\| \leq \|v(x) - v^j(x)\| + \|v^j(x)\| \leq \epsilon, \quad (x \in D_\epsilon),$$

as well as $\mu(D_\epsilon) \geq \mu(Z_j) - \mu(E_j) \geq \delta$.

Let then $F_k := \bigcup_{\ell=k}^\infty D_{2^{-\ell}}$. From the preceding, we deduce $\|v(x)\| \leq 2^{-k}$ on F_k, and $\mu(F_k) \geq \delta$. Taking $D := \bigcap_{k=0}^\infty F_k$, we then have $\mu(D) \geq \delta$ and $v = 0$ on D. This is in contradiction to $\mu(\{x \in A \mid v(x) = 0\}) = 0$. The proof is concluded. □

Proposition 5. *Suppose $u \in \mathrm{BD}(\Omega) \cap L^\infty_{M_u}(\Omega)$ and $I \in \mathrm{BV}(\Omega) \cap L^\infty_{M_I}(\Omega)$. Then $Iu \in \mathrm{BD}(\Omega)$ with*

$$|E(Iu)|(\Omega) \leq M_I |Eu|(\Omega) + M_u |DI|(\Omega).$$

Proof. The proof is similar to the initial parts of the proof of the BV chain rule [20, Theorem 3.96]. Firstly, that $Iu \in L^1(\Omega)$ is obvious from both I and u being L^1 and bounded on Ω. To bound the total deformation $|E(Iu)|(\Omega)$, we take C^1 approximations $u^i \to u$ and $I^i \to I$ strongly in L^1 with $|Eu^i|(\Omega) \to |Eu|(\Omega)$ and $|DI^i|(\Omega) \to |DI|(\Omega)$. Then

$$\begin{aligned}
\mathcal{E}(I^i u^i) &= \frac{1}{2}\left[\nabla(I^i u^i) + (\nabla(I^i u^i))^T\right] \\
&= \frac{1}{2}\left[I^i(\nabla u^i) + I^i(\nabla u^i)^T + (\nabla I^i) \otimes u^i + u^i \otimes (\nabla I^i)\right] \\
&= I^i \mathcal{E} u^i + \nabla I^i \odot u^i.
\end{aligned}$$

Now, since $I^i u^i \in C^1(\Omega)$,

$$\begin{aligned}
|E(I^i u^i)|(\Omega) = \int_\Omega |\mathcal{E}(I^i u^i)|\, dx &\leq \|I^i\|_{L^\infty} \int_\Omega |\mathcal{E} u^i|\, dx + \|u^i\|_{L^\infty} \int_\Omega |\nabla I^i|\, dx \\
&\leq M_I |Eu^i|(\Omega) + M_u |DI^i|(\Omega).
\end{aligned}$$

By the lower semicontinuity of the total variation, letting $i \to \infty$, we obtain the claim. □

References

[1] R. Temam, Mathematical problems in plasticity, Gauthier-Villars, 1985.

[2] L. Ambrosio, A. Coscia, G. D. Maso, Fine Properties of Functions with Bounded Deformation, Archive for Rational Mechanics and Analysis 139 (1997) 201–238.

[3] R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Inventiones Mathematicae 98 (3) (1989) 511–547.

[4] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Inventiones Mathematicae 258 (2004) 227–260.

[5] F. Bouchut, F. James, S. Mancini, Uniqueness and weak stability for multi-dimensional transport equations with one-sided Lipschitz coefficient, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze-Serie V 4 (1) (2005) 1–26.
[6] A. F. Filippov, Differential equations with discontinuous righthand sides, Kluwer Academic Publishers, 1988.
[7] G. Aubert, P. Kornprobst, Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations, Springer, 2nd edn., 2006.
[8] G. Bellettini, A. Coscia, G. Dal Maso, Compactness and lower semicontinuity properties in SBD(Ω), Mathematische Zeitschrift 228 (2) (1998) 337–351.
[9] A. Chambolle, An approximation result for special functions with bounded deformation, Journal de mathématiques pures et appliquées 83 (7) (2004) 929–954.
[10] A. Chambolle, Addendum to "An approximation result for special functions with bounded deformation": the N-dimensional case, Journal de mathématiques pures et appliquées 84 (2005) 137–145.
[11] G. Cortesani, R. Toader, A density result in SBV with respect to non-isotropic energies, Nonlinear Analysis 38 (5) (1999) 585–604.
[12] M. Negri, A non-local approximation of free discontinuity problems in SBV and SBD, Calculus of Variations and Partial Differential Equations 25 (1) (2006) 33–62.
[13] L. Ambrosio, G. Crippa, S. Maniglia, Traces and fine properties of a BD class of vector fields and applications, Annales de la faculté des sciences de Toulouse, Sér. 6 14 (6) (2005) 527–561.
[14] L. Ambrosio, C. de Lellis, J. Malý, On the chain rule for the divergence of BV like vector fields: Applications, partial results, open problems, in: Perspectives in Nonlinear Partial Differential Equations: In Honor of Haim Brezis, vol. 446 of *Contemporary Mathematics*, Americal Mathematical Society, 31–68, 2007.
[15] A. Borzì, K. Ito, K. Kunisch, Optimal control formulation for determining optical flow, SIAM Journal on Scientific Computation 24 (2003) 818.
[16] W. Hinterberger, O. Scherzer, C. Schnorr, J. Weickert, Analysis of optical flow models in the framework of the calculus of variations, Numerical Functional Analysis and Optimization 23 (1-2) (2002) 69–90.
[17] G. Aubert, P. Kornprobst, A Mathematical Study of the Relaxed Optical Flow Problem in the Space BV(Ω), SIAM Journal on Mathematical Analysis 30 (6) (1999) 1282–1308.
[18] S. Keeling, W. Ring, Medical image registration and interpolation by optical flow with maximal rigidity, Journal of Mathematical Imaging and Vision 23 (1) (2005) 47–65.
[19] R. T. Rockafellar, R. J.-B. Wets, Variational Analysis, Springer, ISBN 3540627723, 1998.
[20] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, 2000.
[21] I. Fonseca, G. Leoni, Modern methods in the calculus of variations: L^p spaces, Springer Verlag, 2007.
[22] H. Federer, Geometric Measure Theory, Springer, 1969.
[23] P. Mattila, Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability, Cambridge University Press, 1999.
[24] L. C. Evans, Partial Differential Equations, Americal Mathematical Society, 1998.
[25] G. Crippa, The flow associated to weakly differentiable vector fields, Ph.D thesis, Scuola Normale Superiore di Pisa & Universität Zürich, 2007.