Strong polyhedral approximation of simple jump sets

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Abstract

We prove a strong approximation result for functions $u \in W^{1,\infty}(\Omega \setminus J)$, where J is the union of finitely many Lipschitz graphs satisfying some further technical assumptions. We approximate J by a polyhedral set in such a manner that a regularisation term $\eta(\operatorname{Div}^j u^i)$, $(i = 0, 1, 2, \ldots)$, is convergent. The boundedness of this regularisation functional itself, introduced in [T. Valkonen: "Transport equation and image interpolation with SBD velocity fields", (2011)] ensures the convergence in total variation of the jump part $\operatorname{Div}^j u^i$ of the distributional divergence.

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1. Introduction

Let $u \in \mathrm{SBV}(\Omega)$ be a special function of bounded variation on the domain $\Omega \subset \mathbb{R}^m$. We would like to approximate u by a sequence of functions $\{u^i\}_{i=0}^{\infty}$ such that u^i is reasonably smooth in $\Omega \setminus \widehat{J}_{u^i}$, $(i = 0, 1, 2, \ldots)$, and $\widehat{J}_{u^i}$ is a polyhedral $(m-1)$-dimensional set, containing the jump set J_{u^i}. As the novelty of our results, we would like convergence from a regularisation term $\eta(\operatorname{Div}^j u^i)$, introduced in [11]. The boundedness of this term ensures that if $\operatorname{Div}^j u^i \overset{*}{\rightharpoonup} \operatorname{Div}^j u$ and $|\operatorname{Div}^j u^i| \overset{*}{\rightharpoonup} \lambda$, then $\lambda = |\operatorname{Div}^j u|$. The notation $\operatorname{Div}^j u$ here stands for the "jump part" of the distributional divergence $\operatorname{Div} u$, while the absolutely continuous part will be denoted by $\operatorname{div} u$.

Why do we want this kind of strong approximation property? In [11] we studied an extension of the transport equation involving "jump sources and sinks". With $u = (1, b)$ the velocity field and I the space-time data being transported, it can be stated as

$$\operatorname{Div}(Iu) - I \operatorname{div} u - \tau \operatorname{Div}^j u = 0 \tag{1.1}$$

for some τ defined on the jump set of u, modelling the sources and sinks. To show the stability of (1.1) with $\{I^i\}_{i=0}^{\infty}$ converging weakly in $\mathrm{BV}(\Omega)$ and $\{u^i\}_{i=0}^{\infty}$ converging as in the SBV/SBD compactness theorems [3, 4], we needed to further assume that $|\operatorname{Div}^j u^i|(\Omega) \to |\operatorname{Div}^j u|(\Omega)$. To use (1.1) as a constraint in an optimisation problem (specifically, image interpolation), we thus had to introduce the regularisation term $\eta(\operatorname{Div}^j u^i)$ ensuring this convergence. One possibility for the definition is

$$\eta(\mu) := \sum_{\ell=0}^{\infty} \left(|\mu|(\Omega) - 2^{-\ell m} \int_{\mathbb{R}^m} |\mu(x + [0, 2^{-\ell}]^m)|\, dx \right), \quad (\mu \in \mathcal{M}(\Omega)). \tag{1.2}$$

Roughly $\eta(\mu) < \infty$ says that on average the differences $2^{-\ell m}(|\mu|(x + [0, 2^{-\ell}]^m) - |\mu(x + [0, 2^{-\ell}]^m)|)$ go to zero as the scale $2^{-\ell}$ becomes smaller. Thus on small sets $|\mu|$ is close to μ.

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The problem then becomes: can we, at least in principle, numerically solve problems involving such regularisation terms? That is, can we in particular construct a sequence of discretisations of u such that $\eta(\operatorname{Div}^j u^i) \to \eta(\operatorname{Div}^j u)$ along with the standard convergences $u^i \to u$ and $\nabla u^i \to \nabla u$ in L^2, $D^j u^i \to D^j u$ weakly*, and $\mathcal{H}^{m-1}(J_{u^i}) \to \mathcal{H}^{m-1}(J_u)$? In the present work, we intend to provide a partial answer. Specifically, we restrict our attention to functions $u \in W^{1,\infty}(\Omega \setminus \widehat{J}_u)$, where $\widehat{J}_u$ is the union of finitely many Lipschitz graphs with bounded variation gradient mapping, satisfying further technical conditions, given in Definition 5.1 below. Assuming these conditions, we show that u can be approximated by functions $u^i \in W^{1,\infty}(\Omega \setminus \widehat{J}_{u^i})$ with $\widehat{J}_{u^i}$ polyhedral and satisfying Definition 5.1. Some of our proof techniques resemble those of the SBD approximation theorem of Chambolle [6, 7]. In SBV a counterpart approximation theorem is proved by quite different techniques by Cortesani and Toader [8]. Their result provides largely similar convergence properties as ours, but is missing the crucial convergence of $\eta(\operatorname{Div}^j u^i)$. Of course, the class of functions that we are able to study at the moment is significantly smaller. Finally, we also study anisotropic approximation with $\widehat{J}_{u^i}$ restricted to lie on translations of the coordinate planes.

We have organised this paper as follows. First, in Section 2, we introduce notation and some other well-known tools. In section 3 we study the functional η, and estimates for bounding it. As a consequence we also obtain some new SBV compactness results. In Section 4 we provide a series of further technical lemmas of general nature, needed to prove the approximation theorem. In the subsequent Section 5 we then introduce in detail the space where the approximated function u lies in, and provide further technical lemmas regarding the covering of the boundary of the jump set by cubes. Our main approximation theorem is then stated and proved in Section 6. Finally, we study anisotropic approximation in Section 7.

2. Preliminaries

2.1. Sets and functions

We denote the unit sphere in $\mathbb{R}^m$ by S^{m-1}, while the open ball of radius ρ centred at $x \in \mathbb{R}^m$ we denote by $B(x, \rho)$. The boundary of a set A is denoted ∂A, and the closure by $\operatorname{cl} A$.

For $\nu \in \mathbb{R}^m$, the hyperplane orthogonal to ν we denote by $\nu^\perp := \{z \in \mathbb{R}^m \mid \langle \nu, z\rangle = 0\}$. P_ν denotes the projection onto the subspace spanned by ν, and $P_\nu^\perp$ the projection onto $\nu^\perp$.

We denote by $\{e_1, \ldots, e_m\}$ the standard basis of $\mathbb{R}^m$.

The k-dimensional Jacobian of a linear map $L : \mathbb{R}^k \to \mathbb{R}^m$, $(k \le m)$, is defined as $\mathcal{J}_k[L] := \sqrt{\det(L^* \circ L)}$.

A set $\Gamma \subset \mathbb{R}^m$ is a called a Lipschitz d-graph (of Lipschitz factor L), if there exist a unit vector z_Γ, an open set V_Γ on a d-dimensional subspace of $z_\Gamma^\perp$, and a Lipschitz map $g_\Gamma : V_\Gamma \to \mathbb{R}^m$ of Lipschitz factor at most L, such that

$$\Gamma = \{y \in \mathbb{R}^m \mid g_\Gamma(v) = y,\, v = P_{z_\Gamma}^\perp y \in V_\Gamma\}.$$

We say that Γ is *polyhedral* if g_Γ is piecewise affine and V_Γ is a polyhedral set, i.e., consists of finitely many simplices. If g_Γ is further affine, we say that Γ is *affine*. We define the boundary as $\partial\Gamma := g_\Gamma(\partial V_\Gamma)$.

Remark 2.1. Consider the situation $d = m - 1$. If Γ is the graph of $f : U \subset \mathbb{R}^{m-1} \to \mathbb{R}$, then $g_\Gamma(v) = (x, f(x))$ for $v = (x, 0) \in V_\Gamma = U \times \{0\}$. More generally, if $V_\Gamma \subset z_\Gamma^\perp$ for some $z_\Gamma \in \mathbb{R}^m$, and $f : V_\Gamma \to \mathbb{R}$ is Lipschitz map, then $g_\Gamma(v) = v + z_\Gamma f(v)$ defines a Lipschitz graph. Conversely, if Γ is a Lipschitz graph per the above definition, then defining $f_\Gamma(v) := \langle g_\Gamma(v), z_\Gamma\rangle$ for $v \in V_\Gamma$, we obtain the more conventional description

$$\Gamma = \{v + f_\Gamma(v) z_\Gamma \mid v \in V_\Gamma\}.$$

For our purposes it is more convenient to work with the map g_Γ, however.

2.2. Measures

The space of (signed) Radon measures on an open set Ω is denoted $\mathcal{M}(\Omega)$. If V is a vector space, then the space of V-valued Radon measures on Ω is denoted $\mathcal{M}(\Omega; V)$. The k-dimensional Hausdorff measure, on any given ambient space $\mathbb{R}^m$, $(k \le m)$, is denoted by $\mathcal{H}^k$, while $\mathcal{L}^m$ denotes the Lebesgue measure on $\mathbb{R}^m$. For a measure μ and a measurable set A, we denote by $\mu \llcorner A$ the restriction measure defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$. The total variation measure of μ is denoted $|\mu|$. For a Borel map $u : \Omega \to \mathbb{R}$ we denote $\mu(u) := \int_\Omega u \, d\mu$.

A measure $\mu \in \mathcal{M}(\Omega)$ is said to be *Ahlfors-regular* (in dimension d), if there exists $M \in (0, \infty)$ such that

$$M^{-1} r^d \le |\mu|(B(x, r)) \le M r^d \quad \text{for all} \quad r > 0 \text{ and } x \in \operatorname{supp} \mu.$$

If only the first or the second inequality holds, then μ is said to be, respectively, *lower* or *upper* Ahlfors-regular.

We will often refer to the following standard result on weak* convergence. (See, e.g., [2, Proposition 1.62]).

Proposition 2.1. *Let $\mu^i \in \mathcal{M}(\Omega)$, $(i = 0, 1, 2, \ldots)$, be such that $\mu^i \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega)$, and $|\mu^i| \overset{*}{\rightharpoonup} \lambda \in \mathcal{M}(\Omega)$. If E is a relatively compact μ-measurable set such that $\lambda(\partial E) = 0$, then $\mu^i(E) \to \mu(E)$. More generally, let $u : \Omega \to \mathbb{R}$ be any compactly supported Borel function, and denote by E_f the set of its discontinuity points. Then, if $\lambda(E_f) = 0$, we have $\int_\Omega u \, d\mu^i \to \int_\Omega u \, d\mu$.*

2.3. Functions of bounded variation

A function $u : \Omega \to \mathbb{R}^K$ on a bounded open set $\Omega \subset \mathbb{R}^m$, is said to be of *bounded variation* (see, e.g., [3] for a more thorough introduction), denoted $u \in \mathrm{BV}(\Omega; \mathbb{R}^K)$, if $u \in L^1(\Omega; \mathbb{R}^K)$, and the distributional gradient Du is a Radon measure. We define the norm $\|u\|_{\mathrm{BV}(\Omega;\mathbb{R}^K)} := \|u\|_{L^1(\Omega;\mathbb{R}^K)} + |Du|(\Omega)$.

Given a sequence $\{u^i\}_{i=1}^\infty \subset \mathrm{BV}(\Omega; \mathbb{R}^K)$, strong convergence to $u \in \mathrm{BV}(\Omega; \mathbb{R}^K)$ is defined as strong L^1 convergence $\|u^i - u\|_{L^1(\Omega;\mathbb{R}^K)} \to 0$ together with convergence of the total variation $|u - u^i|(\Omega) \to 0$. Weak convergence is defined as $u^i \to u$ strongly in $L^1(\Omega; \mathbb{R}^K)$ along with $Du^i \overset{*}{\rightharpoonup} Du$ weakly* in $\mathcal{M}(\Omega; \mathbb{R}^{K \times m})$.

We denote by S_u the approximate discontinuity set, i.e., the complement of the set where the Lebesgue limit $\widetilde{u}$ exists. The latter is, of course, defined by

$$\lim_{\rho \searrow 0} \frac{1}{\rho^m} \int_{B(x,\rho)} \|\widetilde{u}(x) - u(y)\| \, dy = 0.$$

The distributional gradient can be decomposed as $Du = \nabla u \mathcal{L}^m + D^j u + D^c u$, where the density ∇u of the *absolutely continuous part* of Du equals (a.e.) the approximate differential of u. The *jump part* $D^j u$ may be represented as

$$D^j u = (u^+ - u^-) \otimes \nu_{J_u} \mathcal{H}^{m-1} \llcorner J_u, \tag{2.1}$$

where x is in the *jump set* $J_u \subset S_u$ of u if for some $\nu := \nu_{J_u}(x)$ there exist two *distinct* one-sided traces $u^\pm(x)$ defined as satisfying

$$\lim_{\rho \searrow 0} \frac{1}{\rho^m} \int_{B^\pm(x,\rho,\nu)} \|u^\pm(x) - u(y)\| \, dy = 0, \tag{2.2}$$

where $B^{\pm}(x,\rho,\nu) := \{y \in B(x,\rho) \mid \pm\langle y-x,\nu\rangle \geq 0\}$. It turns out that J_u is countably $\mathcal{H}^{m-1}$-rectifiable, and ν is (a.e.) the normal to J_u. Moreover, $\mathcal{H}^{m-1}(S_u \setminus J_u) = 0$. The remaining *Cantor part* $D^c u$ vanishes on any Borel set σ-finite with respect to $\mathcal{H}^{m-1}$.

The space $\mathrm{SBV}(\Omega;\mathbb{R}^K)$ of *special functions of bounded variation* is defined as those $u \in \mathrm{BV}(\Omega;\mathbb{R}^K)$ with $D^c u = 0$. There we have the following compactness result.

Theorem 2.1 (SBV compactness [1])**.** *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$. Suppose $\{u^i\}_{i=0}^{\infty} \subset \mathrm{SBV}(\Omega;\mathbb{R}^K)$ with*

$$\sup_i \Big(\|u^i\|_{L^1} + \int_\Omega \psi(|\nabla u^i|)\,dx + |D^j u^i|(\Omega) + \mathcal{H}^{m-1}(J_{u^i}) \Big) < \infty.$$

Then there exists $u \in \mathrm{SBV}(\Omega;\mathbb{R}^K)$ and a subsequence of $\{u^i\}_{i=0}^{\infty}$, unrelabelled, such that

$$\begin{gathered} u^i \to u \text{ strongly in } L^1(\Omega;\mathbb{R}^K), \\ \nabla u^i \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega;\mathbb{R}^{K\times m}), \\ D^j u^i \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(\Omega;\mathbb{R}^{K\times m}), \text{ and} \\ \mathcal{H}^{m-1}(J_u) \leq \liminf_{i\to\infty} \mathcal{H}^{m-1}(J_{u^i}). \end{gathered}$$

We will also be working with functions that are of bounded variation on a subspace. That is, let $z \in S^{m-1}$, and $V \subset z^\perp$ be open and bounded. We then denote $u \in \mathrm{BV}(V;\mathbb{R}^K)$ if $u \circ R_z \in \mathrm{BV}(R_z^{-1}V;\mathbb{R}^K)$, where $R_z \in \mathbb{R}^{m\times(m-1)}$ is an orthonormal basis matrix for $z^\perp$. We let

$$\|u\|_{\mathrm{BV}(V;\mathbb{R}^K)} := \|u \circ R_z\|_{\mathrm{BV}(R_z^{-1}V;\mathbb{R}^K)}.$$

We define the Sobolev spaces $W^{n,p}(V;\mathbb{R}^K)$, $(n \geq 0,\ 1 \leq p \leq \infty)$, analogously.

We are also interested in the case when u has not just scalar or simple vector values, but $u = \nabla g \in L^1(V;\mathbb{R}^K \times z^\perp)$. Then the definition becomes that $u \in \mathrm{BV}(V;\mathbb{R}^K \times z^\perp)$ if $[x \mapsto u(R_z(x))R_z] \in \mathrm{BV}(R_z^{-1}V;\mathbb{R}^{K\times(m-1)})$ with

$$\|u\|_{\mathrm{BV}(V;\mathbb{R}^K\times z^\perp)} := \|x \mapsto u(R_z(x))R_z\|_{\mathrm{BV}(R_z^{-1}V;\mathbb{R}^{K\times(m-1)})}.$$

2.4. Poincaré-type inequalities

We will later need some Poincaré-type inequalities, which we study now. The following proposition can be found in, e.g., [12, Theorem 5.12.7].

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a connected domain with Lipschitz boundary, and μ a positive Radon measure on $\mathbb{R}^d$, that is upper Ahlfors regular with constant M in dimension $d-1$, and satisfies* $\operatorname{supp}\mu \subset \operatorname{cl}\Omega$. *Then there exists a constant $C_1 = C_1(\Omega)$, such that for each $u \in \mathrm{BV}(\Omega)$, we have*

$$\|u - \mu(u)/\mu(\Omega)\|_{L^1(\Omega)} \leq C_1 \frac{M}{\mu(\operatorname{cl}\Omega)} |Du|(\Omega).$$

Corollary 2.1. *Suppose $\Omega = B(0,r)$ in Proposition 2.2. Then there exists a constant $C_2 = C_2(d)$, independent of r, such that*

$$\|u - \mu(u)/\mu(\Omega)\|_{L^1(\Omega)} \leq r^{2d-1} C_2 \frac{M}{\mu(\operatorname{cl}\Omega)} |Du|(\Omega), \quad (u \in \mathrm{BV}(\Omega)). \tag{2.3}$$

Suppose, in particular, that $\mu = \mathcal{L}^d \llcorner \Omega' \subset \Omega$ with $\mu(u) = 0$ and $\mathcal{L}^d(\Omega') \geq \rho r^d$. Then, for a constant $C_3 = C_3(d)$, we have

$$\|u\|_{L^1(\Omega)} \leq r^d \rho^{(1-d)/d} C_3 |Du|(\Omega). \tag{2.4}$$

Proof. We apply Proposition 2.2 on the domain $B(0,1)$ with $u_1(x) := u(rx)$ and $\mu_1(A) := \mu(rA)$, yielding

$$\|u_1 - \mu_1(u_1)/\mu_1(B(0,1))\|_{L^1(B(0,1))} \le C_2 \frac{M_{\mu_1}}{\mu(\operatorname{cl} B(0,1))} |Du_1|(B(0,1)).$$

A change of variables gives

$$|Du_1|(B(0,1)) = |Du|(B(0,r)),$$

and

$$\|u_1 - \mu_1(u_1)/\mu_1(B(0,1))\|_{L^1(B(0,1))} = r^{-d}\|u - \mu(u)/\mu(B(0,r))\|_{L^1(B(0,r))}$$

as $\mu_1(u_1) = \mu(u)$ and $\mu_1(B(0,1)) = \mu(B(0,r))$. Observing that the upper Ahlfors constant M_{μ_1} for μ_1 is at most Mr^{d-1}, we get (2.3).

As for the second result, we just have to approximate M. Elementary manipulations give

$$\mu(B(x,s)) \le \min\{\omega_d s^d, \mathcal{L}^d(\Omega')\} \le Ms^{d-1}$$

for ω_d the volume of the unit ball in $\mathbb{R}^d$, and M defined by

$$M/\mathcal{L}^d(\Omega') = \big(\omega_d/\mathcal{L}^d(\Omega')\big)^{(d-1)/d} \le (\rho^{-1}\omega_d)^{(d-1)/d} r^{1-d}.$$

Inserting this into (2.3) gives (2.4). □

3. Regularisation of total variation

3.1. Convergence of total variation measures

We now study a condition ensuring the convergence of the total variation $|\mu^i|(\Omega)$ subject to the weak* convergence of the measures μ^i, $(i = 0, 1, 2, \ldots)$. Improving a result first presented in [11], we show in Theorem 3.1 below that if $\{f_\ell\}_{\ell=0}^\infty$ is a normalised nested sequence of functions per Definition 3.1 below, then it suffices to bound

$$\eta(\mu^i) := \sum_{\ell=0}^\infty \eta_\ell(\mu^i), \quad \text{where} \quad \eta_\ell(\mu^i) := |\mu^i|(\Omega) - \int |\mu^i(\tau_x f_\ell)|\, dx.$$

Here we employ the notation $\tau_x f(y) := f(y - x)$. In the next subsection we will then study an upper bound on η.

Definition 3.1. Let $f_\ell : \mathbb{R}^m \to \mathbb{R}$, $(\ell = 0, 1, 2, \ldots)$, be bounded Borel functions with compact support that are continuous in $\mathbb{R}^m \setminus S_{f_\ell}$. (That is, the approximate discontinuity set is the discontinuity set.) Let also $\{\nu_\ell\}_{\ell=0}^\infty \subset \mathcal{M}(\mathbb{R}^m)$, $|\nu_\ell|(\mathbb{R}^m) = 1$. The sequence $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ is then said to form a *nested sequence of functions* if $f_\ell(x) = \int f_{\ell+1}(x - y)\, d\nu_\ell(y)$ (a.e.). The sequence is said to be *normalised* if $f_\ell \ge 0$ and $\int f_\ell\, dx = 1$. The sequence is said to be *regular*, if it is normalised, and there exist constants $\alpha > 0$ and $\beta > 0$, and a sequence $h_\ell \searrow 0$,

$$\lim_{r \to 0} \sum_{\ell=0}^\infty \min\{h_\ell, r\} = 0, \tag{3.1}$$

such that $\alpha h_\ell^{-m} \chi_{B(0,\beta h_\ell)} \le f_\ell \le \alpha^{-1} h_\ell^{-m} \chi_{B(0,h_\ell)}$.

Example 3.1. Examples include $f = \chi_{[-1/2,1/2]^m}$ in $\mathbb{R}^m$, and $f(t) = \max\{0, \min\{1 + t, 1 - t\}\}$ in $\mathbb{R}$ (as well as similar but more complicated shape functions in $\mathbb{R}^m$). Regularity holds in these cases, and in the more general typical case $f_\ell(x) := h_\ell^{-m} f(x/h_\ell)$ for $h_\ell \searrow 0$ and some $f \ge \alpha\chi_{B(0,\beta)}$ with compact support and $\int f\, dx = 1$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set, and $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ a normalised nested sequence of functions. Define*

$$\eta(\mu) := \sum_{\ell=0}^{\infty} \eta_\ell(\mu), \quad \textit{where} \quad \eta_\ell(\mu) := |\mu|(\Omega) - \int |\mu(\tau_x f_\ell)|\,dx, \quad (\mu \in \mathcal{M}(\Omega)). \tag{3.2}$$

Suppose $\{\mu^i\}_{i=0}^\infty \subset \mathcal{M}(\Omega)$ weakly converges to $\mu \in \mathcal{M}(\Omega)$ with $\sup_i |\mu^i|(\Omega) + \eta(\mu^i) < \infty$. If also $|\mu^i| \stackrel{*}{\rightharpoonup} \lambda$, then $\lambda = |\mu|$. Moreover, each of the functionals η and η_ℓ, $(\ell = 0, 1, 2, \ldots)$, is lower-semicontinuous with respect to the weak* convergence of $\{\mu^i\}_{i=0}^\infty$. Provided that the weak* convergences hold in $\mathcal{M}(\mathbb{R}^m)$, then also $\eta_\ell(\mu^i) \to \eta_\ell(\mu)$, $(\ell = 0, 1, 2, \ldots)$.*

Proof. Let us suppose first that $\mu^i \stackrel{*}{\rightharpoonup} \mu$ and $|\mu^i| \stackrel{*}{\rightharpoonup} \lambda$ weakly* in $\mathcal{M}(\mathbb{R}^m)$ rather than just $\mathcal{M}(\Omega)$. We denote by E_f the discontinuity set of f, while S_f stands for the approximate discontinuity set. Fubini's theorem and the fact that S_f is an $\mathcal{L}^m$-negligible Borel set, imply that $\int \lambda(S_{\tau_x f_\ell})\,dx = 0$. This shows that $\lambda(S_{\tau_x f_\ell}) = 0$ for a.e. $x \in \mathbb{R}^m$. Since, by assumption $E_f \subset S_f$, it follows that $\lambda(E_{\tau_x f_\ell}) = 0$, so that by Proposition 2.1 we have $\mu^i(\tau_x f_\ell) \to \mu(\tau_x f_\ell)$ for a.e. $x \in \mathbb{R}^m$. Likewise $|\mu^i|(\tau_x f_\ell) \to \lambda(\tau_x f_\ell)$ for a.e. $x \in \mathbb{R}^m$. Since $\sup_i |\mu^i|(\Omega) < \infty$, and Ω is bounded, an application of the dominated convergence theorem now yields

$$\lim_{i\to\infty} \int |\mu^i(\tau_x f_\ell)|\,dx = \int |\mu(\tau_x f_\ell)|\,dx. \tag{3.3}$$

We stress that (3.3) holds because of the convergence $|\mu^i| \stackrel{*}{\rightharpoonup} \lambda$ *in* $\mathcal{M}(\mathbb{R}^m)$ and $\lambda(E_{\tau_x f_\ell}) = 0$.

If we can show that, as claimed, $\lambda = |\mu|$, it follows immediately from (3.3) and the definition (3.2) that $\eta_\ell(\mu^i) \to \eta_\ell(\mu)$, showing that part of the claim of the lemma. Moreover, since the total variation $|\mu^i|(\Omega)$ is lower-semicontinuous with respect to weak* convergence, it follows from (3.3) that each η_ℓ is lower-semicontinuous with respect to the simultaneous weak* convergence of $\{(\mu^i, |\mu^i|)\}_{i=0}^\infty$. Consequently also η is lower-semicontinuous with respect to the simultaneous convergence (by Fatou's lemma). However, assuming that $\{|\mu^i|\}_{i=0}^\infty$ does not converge, let us take a subsequence $\{\mu^{i_n}\}_{n=0}^\infty$ such that $\eta(\mu^{i_n}) \to \alpha := \liminf_{i\to\infty} \eta(\mu^i)$. Since $\sup_i |\mu^i|(\Omega) < \infty$, we may move to a further subsequence, unrelabelled, such that also $|\mu^{i_n}| \stackrel{*}{\rightharpoonup} \lambda$ for some $\lambda \in \mathcal{M}(\Omega)$. Since still $\eta(\mu^{i_n}) \to \alpha$, we deduce from the lower semicontinuity with respect to the simultaneous weak* convergence that $\alpha \geq \eta(\mu)$. This completes the proof of the claim that η is lower-semicontinuous with respect to weak* convergence of $\{\mu^i\}_{i=0}^\infty$ alone.

Returning to the proof of $\lambda = |\mu|$, observe that thanks to the fact that $\{(f_\ell, \nu_\ell)\}_{i=0}^\infty$ is a nested sequence of functions, $\{\eta_\ell(\mu)\}_{\ell=0}^\infty$ forms a decreasing sequence (for any $\mu \in \mathcal{M}(\Omega)$). Indeed, as $f_\ell(x) = \int f_{\ell+1}(x-y)\,d\nu_\ell(y)$ and $\nu_\ell(\mathbb{R}^m) = 1$ with $\nu_\ell \geq 0$, we have

$$\begin{aligned}\int |\mu(\tau_x f_\ell)|\,dx &= \int \left| \int \mu(\tau_{x+y} f_{\ell+1})\,d\nu_\ell(y) \right| dx \leq \int \int |\mu(\tau_{x+y} f_{\ell+1})|\,d\nu_\ell(y)\,dx \\ &= \int \int |\mu(\tau_{x+y} f_{\ell+1})|\,dx\,d\nu_\ell(y) = \int |\mu(\tau_x f_{\ell+1})|\,dx\end{aligned}$$

after a change of variables in the last step to eliminate y. Minding the definition (3.2), it follows from here that $\eta_\ell(\mu) \geq \eta_{\ell+1}(\mu)$.

To show $\lambda = |\mu|$, that is $|\mu^i| \stackrel{*}{\rightharpoonup} |\mu|$, we only have to show $|\mu^i|(\Omega) \to |\mu|(\Omega)$. To see the latter, we choose an arbitrary $\epsilon > 0$, and write

$$|\mu|(\Omega) - |\mu^i|(\Omega) = \eta_\ell(\mu) - \eta_\ell(\mu^i) + \int |\mu(\tau_x f_\ell)| - |\mu^i(\tau_x f_\ell)|\,dx. \tag{3.4}$$

Next we observe from the already proved lower semi-continuity of η and the bound $\sup_i \eta(\mu^i) =: K < \infty$ that $\eta(\mu) \leq K$ as well. Therefore, recalling that $\{\eta_\ell(\mu)\}_{\ell=1}^\infty$ and $\{\eta_\ell(\mu^i)\}_{\ell=1}^\infty$ for $i = 0, 1, \ldots$ are decreasing sequences, as shown above, it follows that by taking j large enough, we can ascertain that $\sup\{\eta_\ell(\mu), \eta_\ell(\mu^1), \eta_\ell(\mu^2), \ldots\} \leq \epsilon$. (Note that $\eta_\ell \geq 0$!) Employing this observation in (3.4), we find that

$$\big||\mu|(\Omega) - |\mu^i|(\Omega)\big| \leq 2\epsilon + \left| \int |\mu(\tau_x f_\ell)| - |\mu^i(\tau_x f_\ell)| \, dx \right|$$

for any large enough j and all i. The integral term tends to zero as $i \to \infty$ by (3.3). Therefore, we have

$$\lim_{i\to\infty} \big||\mu^i|(\Omega) - |\mu|(\Omega)\big| \leq 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this concludes the proof under the assumption that the weak* convergences are in $\mathcal{M}(\mathbb{R}^m)$.

If this assumption does not hold, we may still switch to a subsequence for which $\mu^{i_k} \overset{*}{\rightharpoonup} \bar{\mu}$ and $|\mu^{i_k}| \overset{*}{\rightharpoonup} \bar{\lambda}$ weakly* in $\mathcal{M}(\mathbb{R}^m)$. Then the above reasoning shows that $|\bar{\mu}| = \bar{\lambda}$. But, since Ω is open, necessarily $\bar{\mu}\llcorner\Omega = \mu$ and $\bar{\lambda}\llcorner\Omega = \lambda$. This implies $\lambda = |\mu|$. By the reasoning above, $\eta_\ell(\mu^{i_k}) \to \eta_\ell(\bar{\mu})$. Hence an application of the triangle inequality gives

$$\eta_\ell(\mu) = \eta_\ell(\bar{\mu}\llcorner\Omega) \leq \eta_\ell(\bar{\mu}) = \lim_{k\to\infty} \eta_\ell(\mu^{i_k}).$$

As this bound holds for every subsequence, we deduce that each η_ℓ, ($\ell = 0, 1, 2, \ldots$), is lower-semicontinuous, and consequently η as well. This concludes the proof. □

Remark 3.1. Since, by assumption, $\int f_\ell \, dx = 1$, we may alternatively write $\eta_\ell(\mu) = \int_{\mathbb{R}^m} |\mu|(\tau_x f_\ell) - |\mu(\tau_x f_\ell)| \, dx$.

We will occasionally refer to the following elementary properties that follow from the triangle inequality and the fact that $\operatorname{supp} f_\ell \subset B(0, h_\ell)$.

Lemma 3.1. *Let $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ be a regular nested sequence of functions and $A \subset \mathbb{R}^m$ a Borel set.*

(i) We have

$$\eta_\ell(\mu\llcorner A) + \eta_\ell(\mu\llcorner\mathbb{R}^m \setminus A) \leq \eta_\ell(\mu) \leq \eta_\ell(\mu\llcorner A) + 2|\mu|(\mathbb{R}^m \setminus A).$$

(ii) If $\{\lambda_x\}_{x\in\mathbb{R}^m} \subset \mathcal{M}(\Omega)$, then

$$\int_A |\lambda_x|(\tau_x f_\ell) \, dx \leq \int |\lambda_x \llcorner (A + B(0, h_\ell))|(\tau_x f_\ell).$$

3.2. A bound on geometrical complexity

We now introduce a quantification of the geometrical complexity of a measure or set. It bears some resemblance to definitions of uniform rectifiability, as studied by David and Semmes [9]. That notion, however, does not provide the regularity we need, as it allows considerable "dense" packing of the set, merely measuring locally the deviation from a Lipschitz surface in a geometric sense. Our notion, by contrast, measures the deviation in the sense of measure.

Definition 3.2. Let $\Omega \subset \mathbb{R}^m$ open and bounded, and $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ a *regular* nested sequence of functions per Definition 3.1. Let $\mu \in \mathcal{M}(\Omega)$ be a radon measure, $d \leq m - 1$ and $L, M \in [0, \infty)$. We denote $\mu \in \mathrm{Sp}^d(\Omega, L, M)$ if the following hold.

1. μ is upper Ahlfors-regular in dimension d with constant M.
2. There exist families $\mathcal{G} = \{\mathcal{G}_\ell\}_{\ell=0}^\infty$, $\mathcal{G}_\ell = \{\Gamma_\ell^x \mid x \in \mathbb{R}^m\}$ of d-dimensional Lipschitz graphs Γ_ℓ^x, of Lipschitz factor at most L, satisfying

$$\mathrm{Sp}(\mu; \mathcal{G}) := \sum_{\ell=0}^\infty \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) < \infty, \quad \text{where} \quad \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) := \int_{\mathbb{R}^m} \big|\mu\llcorner O_\ell^x \setminus \Gamma_\ell^x\big|(\tau_x f_\ell) \, dx, \tag{3.5}$$

with the notation $O_\ell^x := x + \operatorname{supp} f_\ell$.

(a) A "simple" set with $\mathrm{Sp}(\Gamma) = \infty$ (b) A "complex" set with $\mathrm{Sp}(\Gamma) < \infty$

Figure 1: Examples of sets satisfying and failing the condition of Definition 3.2.

Definition 3.3. We also set

$$\mathrm{Sp}(\mu) := \inf_{\mathcal{G}} \mathrm{Sp}(\mu; \mathcal{G}), \quad \text{and} \quad \mathrm{Sp}_\ell(\mu) := \inf_{\mathcal{G}_\ell} \mathrm{Sp}(\mu; \mathcal{G}_\ell),$$

where the infimum is taken over all families of the type specified above.

Definition 3.4. For a bounded set $E \subset \mathbb{R}^m$, we denote $E \in \mathrm{Sp}^d(\Omega, L, M)$ if $\mathcal{H}^d \llcorner E \in \mathrm{Sp}^d(\Omega, L, M)$, and set $\mathrm{Sp}_\ell(E; \mathcal{G}) = \mathrm{Sp}_\ell(\mathcal{H}^d \llcorner E; \mathcal{G})$, etc.

Definition 3.5. For the Lipschitz graphs Γ_ℓ^x from Definition 3.2, we use the shorthand notations $V_\ell^x := V_{\Gamma_\ell^x}$, $g_\ell^x := g_{\Gamma_\ell^x}$, and $z_\ell^x := z_{\Gamma_\ell^x}$.

Remark 3.2. Even quite simple sets may fail to satisfy this condition, as Example 3.2 below demonstrates. This poses the question whether this is a reasonable concept. As an element of justification, in Example 3.3 we provide an example of a somewhat "complex" set that satisfies the condition. After that, in Proposition 3.1, we show that the condition implies rectifiability.

Example 3.2. Let us choose $h_\ell := 2^{-\ell}$ and $f_h(x) = h^{-2}\chi_Q(x/h)$ for $Q := [-1/2, 1/2]^2$. We then set $\Gamma_1 = [0,1] \times \{0\}$ and $\Gamma_2 = \{(x, g(x)) \mid x \in [0,1]\}$ for $g(x) = e^{-1/x}$, and study $\mu := \mathcal{H}^1 \llcorner (\Gamma_1 \cup \Gamma_2)$ on $\mathbb{R}^2$. See Figure 1(a) for a sketch.

Suppose $h \in (0,1)$ and let

$$A_h := (h_\ell/2, h_\ell/2) + \{(x,y) \mid x \in [0, 1-h],\, g(x+h) \le h,\, y \in [g(x+h) - h, 0]\}.$$

Then, whenever $(x,y) \in A_h$, both

$$\mathcal{H}^1(\Gamma_i \cap ((x,y) + hQ)) \ge h, \quad (i = 1, 2).$$

Consequently, by the definition of f_h, we find that

$$(\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_h) \ge h^{-1}, \quad (i = 1, 2;\, (x,y) \in A_h).$$

If we set

$$\mathcal{G}_\ell^i := \{(\Gamma_1 \cup \Gamma_2 \setminus \Gamma_i) \cap ((x,y) + h_\ell Q) \mid (x,y) \in \mathbb{R}^2\},$$

we then have

$$h_\ell^{-1} \mathcal{L}^2(A_{h_\ell}) \le \int_{A_{h_\ell}} (\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_{h_\ell})\, d(x,y) \le \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell^i).$$

We want to show that A_h has too large measure for condition (3.5) to be satisfied, that is $h_\ell^{-1}\mathcal{L}^2(A_{h_\ell})$ does not sum to a finite quantity (for any sequence $h_\ell \searrow 0$).

For small enough h, we have

$$A_h \supset \{(x,y) \mid x \ge 0,\, g(x+h) \le h/2,\, y \in [-h/2, 0]\}.$$

Since $g^{-1}(h) = -1/\log h$, we thus have (for small enough h)

$$h^{-1}\mathcal{L}^2(A_h) \geq h^{-1}\int_0^{g^{-1}(h/2)-h} h/2\,dx = (-1/\log(h/2) - h)/2.$$

We observe

$$\sum_{\ell=0}^{\infty}(-1/\log(h_\ell/2) - h_\ell) = \sum_{\ell=0}^{\infty}(1/(\ell+1) - 2^{-\ell}) = \infty.$$

Therefore $\sum_{\ell=0}^{\infty}\mathrm{Sp}_\ell(\mu;\mathcal{G}_\ell^i) = \infty$, $(i = 1, 2)$.

Finally, we observe that there do not exist families $\mathcal{G}_\ell$, $(\ell = 0, 1, 2, \ldots)$, of Lipschitz graphs covering $(\Gamma_1 \cup \Gamma_2) \cap ((x, y) + hQ)$ with bounded constant, so only Γ_1 or Γ_2 can be covered, as has been done above. To see this, one observes that for the Lipschitz constant to be bounded, there must exist $\alpha > 0$ such that any Lipschitz graph Γ covering a part Γ_1 has $|\langle z_\Gamma, (1, 0)\rangle| \geq \alpha$. But then either z_Γ is a tangent vector to Γ_2, or Γ_2 is occluded by Γ_1 when looking in the direction of z. Thus μ fails (3.5).

Example 3.3. Let $r_i := 2^{-i}$, and $\Gamma_i := \{1 - r_i\} \times [0, r_i]$, $(i = 0, 1, 2, \ldots)$. Set then $R := \bigcup_{i=0}^{\infty}\Gamma_i$, as sketched in Figure 1(b). We claim that R satisfies (3.5) with respect to $f_\ell(x) = h_\ell^{-2}\chi_Q(x/h_\ell)$, where $Q := [-1/2, 1/2]^2$. Indeed, at every $(x, y) \in \mathbb{R}^2$, let us choose $\Gamma_\ell^{(x,y)}$ as $\Gamma_i \cap ((x, y) + h_\ell Q)$ for the smallest i such that $1 - r_i \geq x - h_\ell/2$. All we then have to do is to calculate

$$Z_{i,\ell} := \int \mathcal{H}^1 \llcorner (\Gamma_i \setminus \Gamma_\ell^{(x,y)})(\tau_{(x,y)} f_\ell)\,d(x, y), \quad (i = 0, 1, 2, \ldots). \tag{3.6}$$

The term $\mathcal{H}^1 \llcorner (\Gamma_i \setminus \Gamma_\ell^{(x,y)})(\tau_{(x,y)} f_\ell)$ is non-zero only when $x + h_\ell/2 \geq 1 - r_i$ and $x - h_\ell/2 \leq 1 - r_{i-1}$. Minding that $r_{i-1} - r_i = r_i$, it follows that x is on an interval of length $h_\ell - r_i$, and $h_\ell \geq r_i$. For fixed x we may thus calculate that

$$\int (\mathcal{H}^1 \llcorner \Gamma_i)(\tau_{(x,y)} f_\ell)\,dy = h_\ell^{-2}\int\int_y^{y+h_\ell}\chi_{[0,r_i]}(t)\,dt\,dy \leq r_i/h_\ell.$$

This gives the estimate

$$Z_{i,\ell} \leq \begin{cases} (h_\ell - r_i)r_i/h_\ell, & h_\ell \geq r_i, \\ 0, & \text{otherwise}, \end{cases}$$

for the contribution (3.6) of Γ_i, $(i = 0, 1, 2, \ldots)$, to (3.5). But $h_\ell \geq r_i$ means $i \geq -\log_2 h_\ell$, so summing the contributions of Γ_i for $i \geq -\log_2 h_\ell$, we obtain

$$\mathrm{Sp}_\ell(\mu) \leq \sum_{i=0}^{\infty} Z_{i,\ell} \leq \sum_{i \geq -\log_2 h}(h_\ell - r_i)r_i/h_\ell \leq \sum_{i \geq -\log_2 h_\ell} r_i \leq 2h_\ell.$$

Thus (3.5) holds when $\sum_{\ell=0}^{\infty} h_\ell < \infty$. Moreover, it is possible to show that R is Ahlfors-regular in dimension 1, the maximum for the constant M for the upper bound being given at $(1, 0)$.

Proposition 3.1. *Suppose $\Omega \subset \mathbb{R}^m$ is open and bounded, and $\mu \in \mathcal{M}(\Omega)$ satisfies* (3.5). *Then μ is concentrated on a countably d-rectifiable set J. If $\mu \in \mathrm{Sp}^d(\Omega, L, M)$, i.e., μ is also upper Ahlfors-regular, then μ is d-rectifiable, $\mu \ll \mathcal{H}^d \llcorner J$.*

Proof. Let $\mathcal{G}$ be as in Definition 3.2. Let K be a compact set containing $\operatorname{supp}\mu + B(0, h_0)$. To construct rectifiable approximations of $\operatorname{supp}\mu$, we need a partially discrete approximation of the Lebesgue integral over K. Denoting by α and β the regularity constants for $\{f_\ell\}_{\ell=0}^{\infty}$ from Definition 3.1, we set $A_\ell := B(0, \beta h_\ell)$. With ℓ fixed for the moment, we then apply the Besicovitch covering theorem on the family $\{x + A_\ell \mid x \in K\}$ to obtain an at most countable (actually finite) set G_ℓ, such that for a dimensional constant c_m, we have

$$\chi_K \le \sum_{\xi \in G_\ell} \tau_\xi \chi_{A_\ell} \le c_m.$$

It follows that

$$\mathcal{L}^m \ge c_m^{-1} \sum_{\xi \in G_\ell} \mathcal{L}^m \llcorner (\xi + A_\ell). \tag{3.7}$$

Moreover, from the regularity condition for f_ℓ, there exists a constant $C_4 > 0$ dependent on α, β, and m alone, such that

$$\sum_{\xi \in G_\ell} \tau_\xi f_\ell \ge \sum_{\xi \in G_\ell} h_\ell^{-m} \alpha \tau_\xi \chi_{A_\ell} \ge h_\ell^{-m} \alpha \chi_K \ge C_4 / \mathcal{L}^m(A_\ell) \chi_K. \tag{3.8}$$

Now, with this preliminary setup taken care of, let us for any given $y \in A_\ell$ set $J_\ell^y := \bigcup_{x \in G_\ell + y} \Gamma_\ell^x$. Then J_ℓ^y is $\mathcal{H}^d$-rectifiable and we may, using (3.7) and (3.8), approximate

$$\begin{aligned}
\mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) &= \int \big|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\big|(\tau_x f_\ell)\, dx \\
&\ge c_m^{-1} \int_{A_\ell} \sum_{x \in y + G_\ell} \big|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\big|(\tau_x f_\ell)\, dy \\
&\ge c_m^{-1} \int_{A_\ell} \sum_{x \in y + G_\ell} \big|\mu \llcorner \Omega \setminus J_\ell^y\big|(\tau_x f_\ell)\, dy \\
&\ge \frac{C_4}{c_m \mathcal{L}^m(A_\ell)} \int_{A_\ell} \big|\mu \llcorner \Omega \setminus J_\ell^y\big|(\tau_y \chi_K)\, dy \\
&\ge \frac{C_4}{c_m \mathcal{L}^m(A_\ell)} \int_{A_\ell} \big|\mu\big|(\Omega \setminus J_\ell^y)\, dy.
\end{aligned}$$

We thus deduce that there is a choice of $y_\ell \in A_\ell$ with

$$\mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) c_m C^{-1} \ge |\mu|(\Omega \setminus J_\ell^{y_\ell}).$$

Setting $J := \bigcup_{j=0}^\infty J_\ell^{y_\ell}$, it follows from observing

$$|\mu|(\Omega \setminus J_\ell^{y_\ell}) \ge |\mu|(\Omega \setminus J)$$

and letting $\ell \nearrow \infty$ that $|\mu|(\Omega \setminus J) = 0$. Since J is $\mathcal{H}^d$-rectifiable, this gives the first claim of the proposition. If $|\mu|$ is upper Ahlfors-regular in dimension d, we then have $|\mu| \ll \mathcal{H}^d \llcorner J$. We conclude that μ is rectifiable. □

We finish this subsection by showing lower-semicontinuity of the functional $\mu \mapsto \mathrm{Sp}(\mu) + |\mu|(\Omega)$, and, consequently, a closure property of bounded sets in the space $\mathrm{Sp}^d(\Omega, L, M)$.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\{\mu^i\}_{i=0}^\infty \in \mathrm{Sp}^d(\Omega, L, M)$ with*

$$\sup_{i=0,1,2,\ldots} \mathrm{Sp}(\mu^i) + |\mu^i|(\Omega) < \infty.$$

Then any weak limit μ of (a subsequence of) $\{\mu^i\}_{i=0}^\infty$ satisfies $\mu \in \mathrm{Sp}^d(\Omega, L, M)$ and*

$$\mathrm{Sp}(\mu) + |\mu|(\Omega) \le \liminf_{i \to \infty} \mathrm{Sp}(\mu^i) + |\mu^i|(\Omega).$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $\mathcal{G}^i = \{\mathcal{G}^i_\ell\}_{\ell=0}^\infty$, $\mathcal{G}^i_\ell = \{\Gamma_\ell^{x,i} \mid x \in \mathbb{R}^m\}$, be such that

$$\mathrm{Sp}(\mu^i; \mathcal{G}^i) \le \mathrm{Sp}(\mu^i) + \epsilon, \quad (i = 0, 1, 2, \ldots).$$

Then it suffices to show that

$$\mathrm{Sp}(\mu; \mathcal{G}) + |\mu|(\Omega) \le \liminf_{i\to\infty} \mathrm{Sp}(\mu^i; \mathcal{G}^i) + |\mu^i|(\Omega)$$

for some $\mathcal{G} = \{\mathcal{G}_\ell\}_{\ell=0}^\infty$, $\mathcal{G}_\ell = \{\Gamma_\ell^x \mid x \in \mathbb{R}^m\}$.

We use the shorthand notation $z_\ell^{x,i} := z_{\Gamma_\ell^{x,i}}$, and $g_\ell^{x,i} := g_{\Gamma_\ell^{x,i}}$. We may assume that

$$V_{\Gamma_\ell^{x,i}} = P^\perp_{z_\ell^{x,i}} B(x, h_\ell).$$

This is because we may (see, e.g., [10]) extend $g_\ell^{x,i}$ from $V_{\Gamma_\ell^{x,i}}$ to the whole space $(z_\ell^{x,i})^\perp$, without increasing the Lipschitz constant.

We may, moreover, assume that $\mu^i \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega)$, and $|\mu^i| \overset{*}{\rightharpoonup} \lambda \in \mathcal{M}(\Omega)$, where $\lambda \ge |\mu|$. The claim of the proposition now follows by an application of Fatou's inequality in (3.5), if we show for all $\ell = 0, 1, 2, \ldots$ and almost all $x \in \mathbb{R}^m$ that

$$\liminf_{i\to\infty} \left|\mu^i \llcorner O_\ell^x \setminus \Gamma_\ell^{x,i}\right|(\tau_x f_\ell) \ge \left|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\right|(\tau_x f_\ell) \tag{3.9}$$

for some Lipschitz graph Γ_ℓ^x with constant at most L. Indeed, with $\ell = 0, 1, 2, \ldots$ and $x \in \mathbb{R}^m$ fixed, we may for each $i = 0, 1, 2, \ldots$, define a Lipschitz map $g_i : B(0, h_\ell) \subset \mathbb{R}^{m-1} \to \Gamma_\ell^x$ of constant at most L by $g_i(v) = g_\ell^{x,i}(x + R_{z_\ell^{x,i}} v)$ with $R_z \in \mathbb{R}^{m\times(m-1)}$ the basis matrix of $z^\perp$. Then, since Lipschitz maps of bounded constant are compact in the topology of pointwise convergence, we define Γ_ℓ^x as the image of the pointwise limit g of a subsequence of $\{g_i\}_{i=0}^\infty$. Rotating the domain of g back on $z^\perp$ with z a limit of a further subsequence of $\{z_\ell^{x,i}\}_{i=0}^\infty$ will show that Γ_ℓ^x is a Lipschitz graph.

Let us then write

$$\left|\mu^i \llcorner O_\ell^x \setminus \Gamma_\ell^{x,i}\right|(\tau_x f_\ell) = \left|\mu^i\right|(\tau_x f_\ell) - \left|\mu^i \llcorner \Gamma_\ell^{x,i}\right|(\tau_x f_\ell). \tag{3.10}$$

For almost all $x \in \mathbb{R}^m$, we have (as follows from, e.g., [2, Proposition 1.62])

$$|\mu^i|(\tau_x f_\ell) \to \lambda(\tau_x f_\ell). \tag{3.11}$$

Moreover, we have

$$\begin{aligned} \lambda(\tau_x f_\ell) &= (\lambda \llcorner O_\ell^x \setminus \Gamma_\ell^x)(\tau_x f_\ell) + (\lambda \llcorner \Gamma_\ell^x)(\tau_x f_\ell) \\ &\ge \left|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\right|(\tau_x f_\ell) + (\lambda \llcorner \Gamma_\ell^x)(\tau_x f_\ell). \end{aligned} \tag{3.12}$$

On the other hand, any weak* limit $\widetilde{\lambda}$ of (a subsequence of) $|\mu^i| \llcorner \Gamma_\ell^{x,i}$ satisfies $\widetilde{\lambda} \le \lambda \llcorner \Gamma_\ell^x$. Moreover, for a.e. $x \in \mathbb{R}^m$, we have $|\mu^i \llcorner \Gamma_\ell^{x,i}|(\tau_x f_\ell) \to \widetilde{\lambda}(\tau_x f_\ell)$. Thus, minding (3.10)–(3.12), we deduce

$$\begin{aligned} \liminf_{i\to\infty} \left|\mu^i \llcorner O_\ell^x \setminus \Gamma_\ell^{x,i}\right|(\tau_x f_\ell) &= \liminf_{i\to\infty} \Big(\left|\mu^i\right|(\tau_x f_\ell) - \left|\mu^i \llcorner \Gamma_\ell^{x,i}\right|(\tau_x f_\ell)\Big) \\ &\ge \left|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\right|(\tau_x f_\ell) + (\lambda \llcorner \Gamma_\ell^x)(\tau_x f_\ell) - \limsup_{i\to\infty} \left|\mu^i \llcorner \Gamma_\ell^{x,i}\right|(\tau_x f_\ell) \\ &\ge \left|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^{x,i}\right|(\tau_x f_\ell) + (\lambda \llcorner \Gamma_\ell^x)(\tau_x f_\ell) - \widetilde{\lambda}(\tau_x f_\ell) \\ &\ge \left|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\right|(\tau_x f_\ell) \quad \text{for a.e. } x \in \mathbb{R}^m. \end{aligned}$$

But this is (3.9). Since upper Ahlfors regularity clearly holds for μ with constant M by the lower semi-continuity of $|\mu|(B(x, r))$ with respect to weak* convergence, we may conclude the proof. ☐

3.3. Bounds for η

We now intend to derive bounds on $\eta(\mu)$ for measures $\mu \in \mathrm{Sp}^d(\Omega, L, M)$. Throughout we assume that exactly the *same* regular nested sequence of functions $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ is employed in the definition of $\mathrm{Sp}(\mu; \mathcal{G})$ and $\eta(\mu)$. We begin with a technical definition. We need a concept of "bounded variation on a family of Lipschitz surfaces". With this notion we can limit variations in the "intensity" of a rectifiable measure μ, while bounds on $\mathrm{Sp}(\mu; \mathcal{G})$ limit variations in the geometry. Both bounds together then bound $\eta(\mu)$.

Definition 3.6. Suppose θ is a Borel function on a countably $\mathcal{H}^d$-rectifiable set $J \subset \mathbb{R}^m$, and $\mathcal{G}$ a family of Lipschitz d-graphs. We then set

$$\|\theta\|_{\mathrm{BV}(\mathcal{G})} := \sup \sum_{\Gamma_i} \|\theta \circ g_{\Gamma_i}\|_{\mathrm{BV}(V_{\Gamma_i})},$$

where the supremum is taken over all finite *disjoint* sub-collections $\{\Gamma_1, \ldots, \Gamma_N\} \subset \mathcal{G}$, $(N \geq 1)$.

We now state the bounding result. We recall that α and $\{h_\ell\}_{\ell=0}^\infty$ denote regularity constants for the maps $\{f_\ell\}_{\ell=0}^\infty$ from Definition 3.1. Condition (3.13) below is required for uniform constants in Poincaré inequalities; it can trivially be satisfied by extending the domains V_ℓ^x of the Lipschitz graphs Γ_ℓ^x to the whole space $(z_\ell^x)^\perp$, as can be done according to [10].

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $\mu = \theta \mathcal{H}^d \llcorner J \in \mathrm{Sp}^d(\Omega, L, M)$ with $\mathrm{Sp}(\mu; \mathcal{G}) < \infty$ for the collections $\mathcal{G} = \{\mathcal{G}_\ell\}_{\ell=0}^\infty$, $\mathcal{G}_\ell = \{\Gamma_\ell^x \mid x \in \mathbb{R}^m\}$, of Lipschitz graphs of constant at most L. Suppose, moreover, that*

$$\Gamma_\ell^x \cap B(x, h_\ell) \neq \emptyset, \quad \text{and} \quad P_{z_\ell^x}^\perp \Gamma_\ell^x = P_{z_\ell^x}^\perp B(x, h_\ell), \quad (\ell = 0, 1, 2, \ldots;\ x \in \mathbb{R}^m). \tag{3.13}$$

Then

$$\eta_\ell(\mu) \leq C_5 h_\ell^d \|\theta\|_{\mathrm{BV}(\mathcal{G}_\ell)} + \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) \tag{3.14}$$

for some constant $C_5 = C_5(L, m, d, \alpha)$. In particular, if $\sum_{\ell=0}^\infty h_\ell^d < \infty$, then

$$\eta(\mu) \leq C_6 \big(\sup_{\ell=0,1,2,\ldots} \|\theta\|_{\mathrm{BV}(\mathcal{G}_\ell)} + \mathrm{Sp}(\mu; \mathcal{G}) \big)$$

for $C_6 = C_6(L, m, d, \alpha, \sum h_\ell^d)$.

Proof. Let $\ell \in \{0, 1, 2, \ldots\}$ be fixed. By writing $\theta = \theta^+ - \theta^-$, where $\theta^\pm \geq 0$, we deduce

$$\begin{aligned} \eta_\ell(\mu) &= \int |\mu|(\tau_x f_\ell) - |\mu(\tau_x f_\ell)|\, dx \\ &= 2 \int \min \left\{ \int_J \theta^+ \tau_x f_\ell \, d\mathcal{H}^d, \int_J \theta^- \tau_x f_\ell \, d\mathcal{H}^d \right\} dx. \end{aligned} \tag{3.15}$$

Writing $J = (J \cap \Gamma_\ell^x) \cup (J \setminus \Gamma_\ell^x)$, we get

$$\eta_\ell(\mu)/2 \leq \int \min \left\{ \int_{\Gamma_\ell^x} \theta^+ \tau_x f_\ell \, d\mathcal{H}^d, \int_{\Gamma_\ell^x} \theta^- \tau_x f_\ell \, d\mathcal{H}^d \right\} dx + \int \big|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\big| (\tau_x f_\ell)\, dx. \tag{3.16}$$

Since the minimum is non-zero only if both $\theta^+ | O_\ell^x \neq 0$ and $\theta^- | O_\ell^x \neq 0$, only points x in the set

$$Z_\ell := \{x \in \mathbb{R}^m \mid 0 \in \operatorname{conv} \theta(\Gamma_\ell^x),\ \Gamma_\ell^x \cap B(x, h_\ell) \neq \emptyset\}$$

contribute to the first integral in (3.16). Applying (3.5), we thus obtain

$$\begin{aligned} \eta_\ell(\mu)/2 &\le \int_{Z_\ell} \min\left\{ \int_{\Gamma_\ell^x} \theta^+ \tau_x f_\ell \, d\mathcal{H}^d, \int_{\Gamma_\ell^x} \theta^- \tau_x f_\ell \, d\mathcal{H}^d \right\} dx + \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) \\ &\le \alpha^{-1} h_\ell^{-m} \int_{Z_\ell} \min\left\{ \int_{\Gamma_\ell^x} \theta^+ \, d\mathcal{H}^d, \int_{\Gamma_\ell^x} \theta^- \, d\mathcal{H}^d \right\} dx + \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell). \end{aligned} \tag{3.17}$$

In the final step we have used the regularity of $\{f_\ell\}_{\ell=0}^\infty$, i.e., $f_\ell \le \alpha^{-1} h_\ell^{-m} \chi_{B(0,h_\ell)}$.

Next we set $B_\ell := B(0, (2L+4)h_\ell)$, and apply the Besicovitch covering theorem on the family $\{B_\ell + x \mid x \in Z_\ell\}$. With c_m a constant dependent on the dimension m alone, we thus find finite collections $F_\ell^1, \ldots, F_\ell^{c_m} \subset Z_\ell$ satisfying $\sum_{x \in F_\ell^i} \tau_x \chi_{B_\ell} \le 1$, $(i = 1, \ldots, c_m)$, and $\sum_{x \in F_\ell} \tau_x \chi_{B_\ell} \ge \chi_{Z_\ell}$ with $F_\ell := \bigcup_{i=1}^{c_m} F_\ell^i$. Applying the cover $F_\ell + B_\ell$ of Z_ℓ in (3.17), and denoting $\Gamma_\ell^x(\theta) = \int_{\Gamma_\ell^x} \theta \, d\mathcal{H}^d$, we may write

$$\begin{aligned} \eta_\ell(\mu)/2 &\le \alpha^{-1} h_\ell^{-m} \int_{B_\ell} \sum_{x \in (F_\ell + y) \cap Z_\ell} \min\{\Gamma_\ell^x(\theta^+), \Gamma_\ell^x(\theta^-)\} \, dy + \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) \\ &\le \frac{C_7}{\mathcal{L}^m(B_\ell)} \int_{B_\ell} \sum_{x \in (F_\ell + y) \cap Z_\ell} \min\{\Gamma_\ell^x(\theta^+), \Gamma_\ell^x(\theta^-)\} \, dy + \mathrm{Sp}_\ell(\mu; \mathcal{G}_\ell) \end{aligned} \tag{3.18}$$

for some constant $C_7 = C_7(\alpha, m, L)$. By the definition of F_ℓ as $\bigcup_{i=1}^{c_m} F_\ell^i$, it follows that to bound $\eta_\ell(\mu)$, it suffices to show that there exists $C_8 = C_8(d, L)$ such that

$$\sum_{x \in (F_\ell^i + y) \cap Z_\ell} \min\{\Gamma_\ell^x(\theta^+), \Gamma_\ell^x(\theta^-)\} \le C_8 h_\ell^d \|\theta\|_{\mathrm{BV}(\mathcal{G}_\ell)} \tag{3.19}$$

for $\mathcal{L}^m$-a.e. $y \in B_\ell$ and all $i \in \{1, \ldots, c_m\}$.

To begin the proof of (3.19), we observe that $\mathcal{J}_d(\nabla g_\ell^x(v)) \le C_9$ for some $C_9 = C_9(m, d, L)$. This is due to the continuity of $\mathcal{J}_d$ and the bound $\|\nabla g_\ell^x(v)\| \le L$. Thus the area formula yields

$$\Gamma_\ell^x(\theta^\pm) = \int_{\Gamma_\ell^x} \theta^\pm \, d\mathcal{H}^d = \int_{V_\ell^x} (\theta^\pm \circ g_\ell^x) \mathcal{J}_d(\nabla g_\ell^x) \, dv \le C_9 \int_{V_\ell^x} \theta^\pm \circ g_\ell^x \, dv. \tag{3.20}$$

Let us momentarily fix $x \in Z_\ell$, and set $V = V_\ell^x$, $\widetilde{\theta}^\pm = \theta^\pm \circ g_\ell^x$, $z = z_\ell^x$, and $\widetilde{\theta} = \theta \circ g_\ell^x$. We intend to apply Corollary 2.1. Towards this end, we set $\mu^{(\pm)} := \mathcal{L}^d \llcorner (V \setminus \operatorname{supp} \widetilde{\theta}^\mp)$. Then $\mu^{(+)}(V) + \mu^{(-)}(V) \ge \mathcal{L}^d(V)$, so minding (3.13), we have

$$\max\{\mu^{(+)}(V), \mu^{(-)}(V)\} \ge \mathcal{L}^d(V)/2 = \mathcal{L}^d(P_z^\perp B(x, h_\ell))/2 = h_\ell^d \mathcal{L}^d(B(0,1))/2.$$

Since $\mu^{(\pm)}(\widetilde{\theta}^\pm) = 0$, we may apply Corollary 2.1 to get either

$$\|\widetilde{\theta}^+\|_{L^1(V)} \le h_\ell^d C_{10} \|\widetilde{\theta}^+\|_{\mathrm{BV}(V)} \quad \text{or} \quad \|\widetilde{\theta}^-\|_{L^1(V)} \le h_\ell^d C_{10} \|\widetilde{\theta}^-\|_{\mathrm{BV}(V)}$$

for a constant $C_{10} = C_{10}(d)$. As $\|\widetilde{\theta}^\pm\|_{\mathrm{BV}(V)} \le \|\widetilde{\theta}\|_{\mathrm{BV}(V)}$, by the definition of $\theta^\pm$, this gives

$$\min\{\|\widetilde{\theta}^+\|_{L^1(V)}, \|\widetilde{\theta}^-\|_{L^1(V)}\} \le h_\ell^d C_{10} \|\widetilde{\theta}\|_{\mathrm{BV}(V)}.$$

That is

$$\min\{\|\theta^+ \circ g_\ell^x\|_{L^1(V_\ell^x)}, \|\theta^- \circ g_\ell^x\|_{L^1(V_\ell^x)}\} \le h_\ell^d C_{10} \|\theta \circ g_\ell^x\|_{\mathrm{BV}(V_\ell^x)}. \tag{3.21}$$

Next, we observe that with all $\ell \in \{0, 1, 2, \ldots\}$, $i \in \{1, \ldots, c_m\}$, and $y \in B_\ell$ fixed, the graphs $\{\Gamma_\ell^x \mid x \in (y + F_\ell^i \cap Z_\ell)\}$ are disjoint. This follows from the balls $x + B_\ell$, $(x \in y + F_\ell^i)$, being disjoint by construction, and from $\Gamma_\ell^x \subset x + B_\ell = B(x, (2L+4)h_\ell)$. The latter holds due to assumption (3.13) and g_ℓ^x having Lipschitz factor at most L. Combining (3.21) with (3.20) thus finally yields

$$\begin{aligned}\sum_{x\in(F_\ell^i+y)\cap Z_\ell} \min\{\Gamma_\ell^x(\theta^+), \Gamma_\ell^x(\theta^-)\} &\le C_9 C_{10} h_\ell^d \sum_{x\in(F_\ell^i+y)\cap Z_\ell} \|\theta \circ g_\ell^x\|_{\mathrm{BV}(V_\ell^x)} \\ &\le C_9 C_{10} h_\ell^d \|\theta\|_{\mathrm{BV}(\mathcal{G}_\ell)}.\end{aligned} \tag{3.22}$$

To conclude the proof of the proposition, we only have to observe that (3.22) yields (3.19). □

Remark 3.3. Let $\{(\widetilde{f}_\ell, \widetilde{\nu}_\ell)\}_{\ell=0}^\infty$ be another nested sequence of functions that satisfies $f_\ell \le C\widetilde{f}_\ell$ for some $C > 0$. Then in (3.16) we could approximate

$$\int \big|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\big|(\tau_x f_\ell)\,dx \le \int C\big|\mu \llcorner O_\ell^x \setminus \Gamma_\ell^x\big|(\tau_x \widetilde{f}_\ell)\,dx \le C\widetilde{\mathrm{Sp}}_\ell(\mu; \mathcal{G}_\ell),$$

where $\widetilde{\mathrm{Sp}}_\ell$ denotes the functional Sp_ℓ obtained with the sequence $\{(\widetilde{f}_\ell, \widetilde{\nu}_\ell)\}_{\ell=0}^\infty$. Thus it would, at the expense of additional technical complexity that we want to avoid, be possible to express our results for different sequences of nested functions for the definitions of η and Sp.

3.4. Compactness in $\mathrm{SBV}(\Omega; \mathbb{R}^K)$

We finish this section by providing some compactness results in $\mathrm{SBV}(\Omega; \mathbb{R}^K)$ following immediately from the results above. They can be useful in applications for proving closure properties. We need to work with vector-valued measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{K\times m})$. The results above on $\mathrm{Sp}(\mu)$ can readily be extended to this situation with no changes in proofs or definitions, but for concreteness we work through the following definition.

Definition 3.7. For $\mu = (\mu_{i,n}) \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$, we denote $\mathrm{Sp}(\mu) = \sum_{i=1}^K \sum_{n=1}^m \mathrm{Sp}(\mu_{i,j})$.

Our main compactness result is then as follows. The difference to the well-established Theorem 2.1 is that we replace $\mathcal{H}^{m-1}(J_{u^i})$ by $\mathrm{Sp}(D^j u^i)$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $\{u^i\}_{i=0}^\infty \subset \mathrm{SBV}(\Omega; \mathbb{R}^K)$. Suppose $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$. If each $D^j u^i \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$, $(i = 0, 1, 2, \ldots)$, and*

$$\sup_i \|u^i\|_{L^1(\Omega)} + \int \psi(\nabla u^i(x))\,dx + |D^j u^i|(\Omega) + \mathrm{Sp}(D^j u^i) < \infty, \tag{3.23}$$

there then exists $u \in \mathrm{SBV}(\Omega; \mathbb{R}^K)$ with $D^j u \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$ and a subsequence, unrelabelled, such that

$$u^i \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^K), \tag{3.24}$$

$$\nabla u^i \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega; \mathbb{R}^{K\times m}), \tag{3.25}$$

$$D^j u^i \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(\Omega; \mathbb{R}^{K\times m}), \quad \text{and} \tag{3.26}$$

$$\mathrm{Sp}(D^j u) \le \liminf_{i\to\infty} \mathrm{Sp}(D^j u^i). \tag{3.27}$$

Proof. Let us denote by K the supremum on the left side of (3.23). We then deduce from (3.23) that

$$\sup_i \|u^i\|_{L^1(\Omega)} + |Du^i|(\Omega) < \infty.$$

Moving to a subsequence, unrelabelled, we may thus assume that $u^i \rightharpoonup u$ weakly in $[\mathrm{BV}(\Omega)]^k$ for some $u \in \mathrm{BV}(\Omega; \mathbb{R}^K)$. This gives (3.24). Moreover, because $\{\nabla u^i\}_{i=0}^\infty$ is an equi-integrable family, we have the existence of some $w \in L^1(\Omega; \mathbb{R}^{K\times m})$, such that for a further unrelabelled subsequence, $\nabla u^i \rightharpoonup w$ weakly in $L^1(\Omega; \mathbb{R}^{K\times m})$. Still, selecting another subsequence, we find from Proposition 3.2 that $D^j u^i \overset{*}{\rightharpoonup} \lambda$ for some $\lambda \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$ with $\mathrm{Sp}(\lambda) \le \liminf_{i\to\infty} \mathrm{Sp}(D^j u^i)$. Minding that $\nabla u^i \mathcal{L}^m + D^j u^i = Du^i$ and $Du^i \overset{*}{\rightharpoonup} Du$ by the weak convergence of $\{u^i\}_{i=0}^\infty$ in $\mathrm{BV}(\Omega; \mathbb{R}^K)$, we therefore have

$$w\mathcal{L}^m + \lambda = Du = \nabla u \mathcal{L}^m + D^j u + D^c u. \tag{3.28}$$

Since $\lambda \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$, Proposition 3.1 shows that the measure λ is concentrated on a $\mathcal{H}^{m-1}$ rectifiable set J. This gives $w = \nabla u$, showing (3.25). According to [2], the Cantor part $D^c u$ vanishes on any Borel set B that is σ-finite with respect to $\mathcal{H}^{m-1}$. In particular $D^c u \llcorner J = 0$. Hence, by (3.28), $\lambda = D^j u$ and $D^c u = 0$. This shows that $u \in \mathrm{SBV}(\Omega; \mathbb{R}^K)$ as well as (3.26) and (3.27), thus completing the proof. □

We now state a corollary that be used to prove the closedness of equations like (1.1). Specifically, we show stronger convergence for $T \circ D^j u^i$ with $T : \mathbb{R}^{K\times m} \to \mathbb{R}$ a bounded linear operator by bounding $\eta(T \circ D^j u^i)$. When $K = m$, choosing $T = \mathrm{Tr}$ as the trace operator, we get the convergence in total variation of the jump part $\mathrm{Div}^j u^i := \mathrm{Tr} \circ D^j u^i$ of the distributional divergence, appearing in (1.1) and more precisely given by

$$\mathrm{Div}^j u^i(\varphi) = (\mathrm{Tr} \circ D^j u^i)(\varphi) = \sum_{n=1}^{m} \langle e_n, D^j u^i(\varphi) e_n\rangle, \quad (\varphi \in C_c(\Omega)).$$

Here $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $\{u^i\}_{i=0}^\infty \subset \mathrm{SBV}(\Omega; \mathbb{R}^K)$. Suppose $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing with $\lim_{t\to\infty} \psi(t)/t = \infty$, and $T : \mathbb{R}^{K\times m} \to \mathbb{R}$ a bounded linear operator. If each $D^j u^i \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$, $(i = 0, 1, 2, \ldots)$, and*

$$\sup_i \|u^i\|_{L^1(\Omega)} + \int \psi(\nabla u^i(x))\, dx + |D^j u^i|(\Omega) + \mathrm{Sp}(D^j u^i) + \eta(T \circ D^j u^i) < \infty, \tag{3.29}$$

then there exists $u \in \mathrm{SBV}(\Omega; \mathbb{R}^K)$ with $D^j u \in [\mathrm{Sp}^{m-1}(\Omega, L, M)]^{K\times m}$, and a subsequence, unrelabelled, such that (3.24)–(3.27) hold along with

$$T \circ D^j u^i \overset{*}{\rightharpoonup} T \circ D^j u \quad \textit{weakly* in } \mathcal{M}(\Omega), \quad \textit{and} \tag{3.30}$$

$$|T \circ D^j u^i|(\Omega) \to |T \circ D^j u|(\Omega). \tag{3.31}$$

Proof. Theorem 3.2 shows that (3.24)–(3.27) hold. As an immediate consequence, we also get (3.30). Now (3.31) follows from Theorem 3.1. □

4. Technical results

We now prove a couple of general technical results that we will be needing in the proof of the approximation theorem. We begin with a result on graph approximation, for which we need the following elementary lemma.

Lemma 4.1. *Let $\Gamma \subset \mathbb{R}^m$ be a Lipschitz $(m-1)$-graph with normal field ν_Γ. Then*

$$(\nu_\Gamma \circ g_\Gamma)(v) = A_\Gamma \nabla g_\Gamma(v)/\|A_\Gamma \nabla g_\Gamma(v)\|, \quad (\textit{a.e. } v \in V_\Gamma),$$

for the linear operator A_Γ defined by

$$A_\Gamma G = (I - H_\Gamma G^*) z_\Gamma,$$

with $H_\Gamma : z_\Gamma^\perp \to \mathbb{R}^m$ *the injection operator and* $G : z_\Gamma^\perp \to \mathbb{R}^m$ *an arbitrary linear operator. Moreover*

$$\|A_\Gamma\| \geq 1, \tag{4.1}$$

and the map defined by

$$F_\Gamma(G) := A_\Gamma G / \max\{1, \|A_\Gamma G\|\}$$

has Lipschitz factor $\mathrm{Lip}(F_\Gamma) = 1$.

Proof. For some $f_\Gamma : z_\Gamma^\perp \to \mathbb{R}$ we have $g_\Gamma(v) = H_\Gamma v + f_\Gamma(v) z_\Gamma$ and

$$\nabla g_\Gamma(v) = H_\Gamma + z_\Gamma \otimes \nabla f_\Gamma(v).$$

We have $H_\Gamma^* z_\Gamma = 0$ and

$$H_\Gamma^* \nabla g_\Gamma(v) = H_\Gamma^* H_\Gamma + H_\Gamma^* z_\Gamma \otimes \nabla f(v) = H_\Gamma^* H_\Gamma = I,$$

so that for any $v' \in z_\Gamma^\perp$, $v \in V_\Gamma$, we get

$$\langle (I - H_\Gamma(\nabla g_\Gamma(v))^*) z_\Gamma, \nabla g_\Gamma(v) v' \rangle = 0.$$

Since the tangent cone $T_\Gamma(g_\Gamma(v)) = \nabla g_\Gamma(v) z_\Gamma^\perp$ a.e., this says that

$$\nu_\Gamma(g_\Gamma(v)) = \frac{(I - H_\Gamma(\nabla g_\Gamma(v))^*) z_\Gamma}{\|(I - H_\Gamma(\nabla g_\Gamma(v))^*) z_\Gamma\|} = \frac{A_\Gamma \nabla g_\Gamma(v)}{\|A_\Gamma \nabla g_\Gamma(v)\|}, \quad (\text{a.e. } v \in V_\Gamma). \tag{4.2}$$

Thanks to $H_\Gamma^* z_\Gamma = 0$, we deduce that

$$\|A_\Gamma\| \geq \|z_\Gamma - H_\Gamma G^* z_\Gamma\| = \sqrt{\|z_\Gamma\|^2 + \|H_\Gamma G^* z_\Gamma\|^2} \geq \|z_\Gamma\| = 1,$$

with $G : z_\Gamma^\perp \to \mathbb{R}^m$ an arbitrary linear operator of norm $\|G\| = 1$. Finally, thanks to $\|F_\Gamma G\| \leq \|A_\Gamma G\|$, we have

$$\|F_\Gamma G_1 - F_\Gamma G_2\| \leq \|A_\Gamma G_1 - A_\Gamma G_2\| = \|H_\Gamma (G_1 - G_2)^* z_\Gamma\| \leq \|G_1 - G_2\|,$$

so that F_Γ is Lipschitz with factor $\mathrm{Lip}(F_\Gamma) = 1$. □

Lemma 4.2. *Let* $\Gamma \Subset \mathbb{R}^m$ *be a Lipschitz* $(m-1)$*-graph with* $\partial\Gamma \subset \mathrm{int}\, \widehat{Z}$ *and* $\mathcal{H}^{m-1}(\partial\widehat{Z} \cap \Gamma) = 0$ *for a closed set* $\widehat{Z}$*. Let* $\{s^k\}_{k=0}^\infty \subset (0, \bar{s})$ *with* $s^k \searrow 0$*,* $(k \to \infty)$*. Suppose that* $\nabla g_\Gamma \in \mathrm{BV}(V_\Gamma; \mathbb{R}^m \times z_\Gamma^\perp)$*. Then we can find polyhedral Lipschitz graphs* $\{\Gamma^k\}_{k=0}^\infty$ *of factor at most* $L' = L'(\Gamma)$*, satisfying* $\partial\Gamma^k \subset \widehat{Z}$*,* $z_{\Gamma^k} = z_\Gamma$*,* $V_{\Gamma^k} \subset V_\Gamma$*,* $(k = 0, 1, 2, \ldots)$*, and*

$$\Gamma^k \subset \Gamma \setminus \widehat{Z} + B(0, s^k/2). \tag{4.3}$$

We also have the convergences

$$\mathcal{H}^{m-1} \llcorner \Gamma^k \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner \Gamma \setminus \widehat{Z} \quad \textit{weakly* in } \mathcal{M}(\mathbb{R}^m), \quad (k \to \infty), \tag{4.4}$$

$$\nu_{\Gamma^k} \mathcal{H}^{m-1} \llcorner \Gamma^k \overset{*}{\rightharpoonup} \nu_\Gamma \mathcal{H}^{m-1} \llcorner \Gamma \setminus \widehat{Z} \quad \textit{weakly* in } \mathcal{M}(\mathbb{R}^m; S^{m-1}), \quad (k \to \infty). \tag{4.5}$$

Regarding the maps $\{g_{\Gamma^k}\}_{k=0}^\infty$*, we have* $\nabla g_{\Gamma^k} \in \mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m \times z_\Gamma^\perp)$ *with*

$$\|g_{\Gamma^k} - g_\Gamma\|_{L^\infty(V_{\Gamma^k}; \mathbb{R}^m)} \leq s^k/2, \tag{4.6}$$

$$\|\nu_{\Gamma^k} \circ g_{\Gamma^k} - \nu_\Gamma \circ g_\Gamma\|_{L^1(V_{\Gamma^k}; \mathbb{R}^m)} \leq s^k, \quad \textit{and} \tag{4.7}$$

$$\|\nu_{\Gamma^k} \circ g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m)} \leq \|\nabla g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m \times z_\Gamma^\perp)} \leq C_{11}\big(\|g_\Gamma\|_{L^1(V_\Gamma; \mathbb{R}^m)} + \|\nabla g_\Gamma\|_{\mathrm{BV}(V_\Gamma; \mathbb{R}^m \times z_\Gamma^\perp)}\big) \tag{4.8}$$

for some constant $C_{11} = C_{11}(m)$.

Proof. Suppose we construct $\Gamma^k := g_{\Gamma^k}(\widetilde{V}^k) \setminus \widehat{Z}$ for some $g_{\Gamma^k} : z_\Gamma^\perp \to \mathbb{R}^m$ of Lipschitz factor at most L', and polyhedral $\widetilde{V}^k \subset V_\Gamma$ with $\Gamma \subset g_\Gamma(\widetilde{V}^k) \cup \widehat{Z}$. Then $z_{\Gamma^k} = z_\Gamma$ and $V_{\Gamma^k} = g_{\Gamma^k}^{-1}(\Gamma^k) \subset \widetilde{V}^k$ with $\partial\Gamma^k \subset \widehat{Z}$ holding. Moreover, (4.3) follows if we show (4.6).

Since $\|\nabla g_{\Gamma^k}(v)\| \geq 1$, $(v \in \widetilde{V}^k)$, we deduce from Lemma 4.1 that $\nu_{\Gamma^k} \circ g_{\Gamma^k} = F_{\Gamma^k} \circ \nabla g_{\Gamma^k}$ for the Lipschitz function F_{Γ^k}. Since $\|\nabla g_{\Gamma^k}(x)\| \geq 1$ and $\|F_{\Gamma_k}(G)\| \leq 1$ for all x, G, we find that

$$\|\nu_{\Gamma^k} \circ g_{\Gamma^k}\|_{L^1(V_{\Gamma^k};\mathbb{R}^m)} = \|F_{\Gamma^k} \circ \nabla g_{\Gamma^k}\|_{L^1(V_{\Gamma^k};\mathbb{R}^m)} \leq \|\nabla g_{\Gamma^k}\|_{L^1(V_{\Gamma^k};\mathbb{R}^m)}.$$

If $\nabla g_{\Gamma^k} \in \mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m \times z_\Gamma^\perp)$, it thus follows from the BV chain rule and $\mathrm{Lip}(F_{\Gamma^k}) = 1$ that $\nu_{\Gamma^k} \circ g_{\Gamma^k} \in \mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m)$ with

$$\begin{aligned}
\|\nu_{\Gamma^k} \circ g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k};\mathbb{R}^m)} &= \|F_{\Gamma^k} \circ \nabla g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k};\mathbb{R}^m)} \\
&= \|F_{\Gamma^k} \circ \nabla g_{\Gamma^k} \circ R_{z_\Gamma}\|_{\mathrm{BV}(R_{z_\Lambda}^{-1} V_{\Gamma^k};\mathbb{R}^m)} \\
&\leq \|x \mapsto \nabla g_{\Gamma^k}(R_{z_\Gamma} x) R_{z_\Gamma}\|_{\mathrm{BV}(R_{z_\Lambda}^{-1} V_{\Gamma^k};\mathbb{R}^{m \times (m-1)})} \\
&= \|\nabla g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k};\mathbb{R}^m \times z_\Gamma^\perp)}.
\end{aligned}$$

From the Lipschitz property of F_{Γ^k}, we also deduce that

$$\begin{aligned}
\|\nu_{\Gamma^k} \circ g_{\Gamma^k} - \nu_\Gamma \circ g_\Gamma\|_{L^1(V_{\Gamma^k};\mathbb{R}^m)} &= \|F_{\Gamma^k} \circ \nabla g_{\Gamma^k} - F_\Gamma \circ \nabla g_\Gamma\|_{L^1(V_{\Gamma^k};\mathbb{R}^m)} \\
&\leq \|\nabla g_{\Gamma^k} - \nabla g_\Gamma\|_{L^1(V_{\Gamma^k};\mathbb{R}^m \times z_\Gamma^\perp)}.
\end{aligned}$$

Thus (4.7) and (4.8) follow from showing

$$\|\nabla g_{\Gamma^k}\|_{\mathrm{BV}(V_{\Gamma^k};\mathbb{R}^m \times z_\Gamma^\perp)} \leq C_{11}\big(\|g_\Gamma\|_{L^1(V_\Gamma;\mathbb{R}^m)} + \|\nabla g_\Gamma\|_{\mathrm{BV}(V_\Gamma;\mathbb{R}^m \times z_\Gamma^\perp)}\big), \tag{4.9}$$

and, respectively,

$$\|\nabla g_{\Gamma^k} - \nabla g_\Gamma\|_{L^1(V_{\Gamma^k};\mathbb{R}^m \times z_\Gamma^\perp)} \leq s^k. \tag{4.10}$$

Next we want to show that (4.4), (4.5) follow if we show (4.6) and (4.10). Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^m)$ and define $U := R_{z_\Gamma}^{-1} \widetilde{V}^k$, as well as $\widetilde{g} = g_\Gamma \circ R_{z_\Gamma}$ and $\widetilde{g}^k = g_{\Gamma^k} \circ R_{z_\Gamma}$, where we recall that $R_z : \mathbb{R}^{m-1} \to z^\perp$ is the basis matrix of $z^\perp$. Then the area formula gives

$$\begin{aligned}
&\int_{g_{\Gamma^k}(\widetilde{V}^k)} \varphi \, d\mathcal{H}^{m-1} - \int_{g_\Gamma(\widetilde{V}^k)} \varphi \, d\mathcal{H}^{m-1} \\
&\qquad = \int_U \varphi(\widetilde{g}^k(x)) \mathcal{J}_{m-1}(\nabla \widetilde{g}^k(x)) \, dx - \int_U \varphi(\widetilde{g}(x)) \mathcal{J}_{m-1}(\nabla \widetilde{g}(x)) \, dx.
\end{aligned}$$

Employing the fact that the map $(x, y) \mapsto xy$ is Lipschitz on bounded sets, it follows that

$$\begin{aligned}
\left| \int_{g_{\Gamma^k}(\widetilde{V}^k)} \varphi \, d\mathcal{H}^{m-1} - \int_{g_\Gamma(\widetilde{V}^k)} \varphi \, d\mathcal{H}^{m-1} \right| &\leq \int_U \left| \varphi(\widetilde{g}^k(x)) \mathcal{J}_{m-1}(\nabla \widetilde{g}^k(x)) - \varphi(\widetilde{g}(x)) \mathcal{J}_{m-1}(\nabla \widetilde{g}(x)) \right| dx \\
&\leq C_{12} \left(\int_U \left| \varphi(\widetilde{g}^k(x)) - \varphi(\widetilde{g}(x)) \right| dx + \int_U \left| \mathcal{J}_{m-1}(\nabla \widetilde{g}^k(x)) - \mathcal{J}_{m-1}(\nabla \widetilde{g}(x)) \right| dx \right)
\end{aligned} \tag{4.11}$$

for some constant $C_{12} = C_{12}(\varphi, L')$. Minding (4.6), the first integral of (4.11) goes to zero because $\varphi \in C_c^\infty(\mathbb{R}^m)$ is uniformly continuous. For the second integral, we observe from (4.10) that $\nabla \widetilde{g}^k$ converges to $\nabla \widetilde{g}$ in L^1, which we recall to imply almost uniform convergence for a subsequence. That is, after possibly switching to an unrelabelled subsequence, for every $\epsilon > 0$ there exists a measurable subset $E \subset \Omega$ with $\mathcal{L}^m(\Omega \setminus E) < \epsilon$, and $\nabla \widetilde{g}^k \to \nabla \widetilde{g}$ uniformly on E. By the uniform Lipschitz continuity of $\{g^k\}_{k=0}^\infty$, the values of $\nabla \widetilde{g}^k$ moreover lie in a bounded set. With these observations it now easily follows that the second integral of (4.11) also tends to zero. Thus the left hand side of (4.11) tends to zero. We have therefore shown that

$$\mathcal{H}^{m-1} \llcorner g_{\Gamma^k}(\widetilde{V}^k) \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner g_\Gamma(V_\Gamma).$$

By assumption $\mathcal{H}^{m-1}(\Gamma \cap \partial \widehat{Z}) = 0$, so that by Proposition 2.1

$$\mathcal{H}^{m-1} \llcorner g_{\Gamma^k}(\widetilde{V}^k) \setminus \widehat{Z} \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner g_\Gamma(V_\Gamma) \setminus \widehat{Z}. \tag{4.12}$$

Minding the construction of Γ^k, we have both

$$\mathcal{H}^{m-1} \llcorner \Gamma^k = \mathcal{H}^{m-1} \llcorner g_{\Gamma^k}(\widetilde{V}^k) \setminus \widehat{Z} \quad \text{and} \quad \mathcal{H}^{m-1} \llcorner g_\Gamma(V_\Gamma) \setminus \widehat{Z} = \mathcal{H}^{m-1} \llcorner \Gamma \setminus \widehat{Z}. \tag{4.13}$$

The convergence (4.4) now follows from (4.12) and (4.13). Since (4.5) can be shown in a similar fashion with the help of (4.7), we skip the details.

It remains to construct g_{Γ^k} and V_{Γ^k} such that (4.6), (4.9), and (4.10) hold. To begin with, let $\{\mathcal{T}_\ell\}_{\ell=0}^\infty$, be a sequence of uniform triangulations of $z_\Gamma^\perp$, each a subdivision of the previous with edge length approaching zero as $\ell \to \infty$. We then let

$$\widetilde{V}_\ell := \bigcup \{T \in \mathcal{T}_\ell \mid T \subset V_\Gamma\}.$$

For sufficiently large ℓ, we have $\Gamma \setminus \widehat{Z} \subset g_\Gamma(\widetilde{V}_\ell)$ and $g_\Gamma(\partial \widetilde{V}_\ell) \subset \operatorname{int} \widehat{Z}$. Since $\nabla g_\Gamma \in \mathrm{BV}(V_\Gamma; \mathbb{R}^m \times z_\Gamma^\perp)$, we may by mollification approximate g_Γ on $\widetilde{V}_\ell$ by smooth functions g_ϵ, satisfying for sufficiently small $\epsilon > 0$ estimates of the type (4.6), (4.10) along with $g_\epsilon(\partial \widetilde{V}^k) \subset \operatorname{int} \widehat{Z}$ and

$$\|\nabla g_\epsilon\|_{\mathrm{BV}(\widetilde{V}^k; \mathbb{R}^m \times z_\Gamma^\perp)} \le \|\nabla g_\Gamma\|_{\mathrm{BV}(V_\Gamma; \mathbb{R}^m \times z_\Gamma^\perp)}.$$

Moreover, the Lipschitz factor of g_ϵ is bounded by that of g_Γ. As a consequence of this approximation, we may assume that

$$g_\Gamma \in W^{1,\infty}(V_\Gamma; \mathbb{R}^m) \cap W^{2,1}(V_\Gamma; \mathbb{R}^m). \tag{4.14}$$

For each $\ell = 0, 1, 2, \ldots$, we denote by $\{x_{\ell,n}\}_{n=1}^{M_\ell}$ the nodal points of the triangulation $\mathcal{T}_\ell$. Define $\varphi_{\ell,n}$ such that it is affine on each T and

$$\operatorname{supp} \varphi_{\ell,n} \subset K_{\ell,n} := \bigcup_{T \in \mathcal{T}_\ell : x_{\ell,n} \in \partial T} T.$$

We then define $g^k : \widetilde{V}^k \to \mathbb{R}^m$ as

$$g^k := \sum_{n=1}^{M_{\ell(k)}} \varphi_{\ell(k),n} g(x_{\ell(k),n}), \quad (k = 0, 1, 2, \ldots)$$

for some $\ell(k) \ge k$. That is, g^k is the Lagrange interpolation of g on $\mathcal{T}_{\ell(k)}$. Minding that we have without loss of generality assumed (4.14), choosing $\ell(k)$ is sufficiently large, we observe that g^k satisfies for some constant $C_{13} = C_{13}(m, \mathcal{T}^1)$ the standard finite element estimates (see, e.g., [5])

$$\begin{gathered} \|g^k\|_{W^{1,\infty}(\widetilde{V}^k; \mathbb{R}^m)} \le C_{13} \|g_\Gamma\|_{W^{1,\infty}(V_\Gamma; \mathbb{R}^m)}, \\ \|g^k - g_\Gamma\|_{L^\infty(\widetilde{V}^k; \mathbb{R}^m)} \le s^k/2, \quad \text{and} \\ \|\nabla g^k - \nabla g_\Gamma\|_{L^1(\widetilde{V}^k; \mathbb{R}^{m \times m})} \le s^k/4, \quad (k = 0, 1, 2, \ldots). \end{gathered}$$

In particular, g^k has Lipschitz factor at most $L'(\Gamma) = C_{13} \|g_\Gamma\|_{W^{1,\infty}(V_\Gamma; \mathbb{R}^m)}$, and (4.6), (4.10) are satisfied. Finally, to show (4.9), we observe that

$$\|\nabla g^k\|_{\mathrm{BV}(V_{\Gamma^k}; \mathbb{R}^m \times z_\Gamma^\perp)} \le C_{14} \|g_\Gamma\|_{W^{2,1}(V_\Gamma; \mathbb{R}^m)}, \quad (k = 0, 1, 2, \ldots), \tag{4.15}$$

for some constant $C_{14} = C_{14}(m, \mathcal{T}^1)$. For piecewise affine shape functions, this does not follow from standard results due to insufficient regularity. If we use smooth (or $W^{2,1}$) shape functions, we however get by standard results (see [5, Theorem 4.5.11]) that

$$\|\nabla g^k\|_{\mathrm{BV}(V_{\Gamma^k};\mathbb{R}^m \times z_\Gamma^\perp)} \le \|g^k\|_{W^{2,1}(V_{\Gamma^k};\mathbb{R}^m)} \le C_{14}\|g_\Gamma\|_{W^{2,1}(V_\Gamma;\mathbb{R}^m)}, \quad (k = 0, 1, 2, \dots).$$

Thus, to get (4.15), we can simply approximate the piecewise affine shape functions by smooth shape functions on the same triangulation $\mathcal{T}^k$ and pass to the limit. (To construct such smooth shape functions, for each $\varphi = \varphi_{\ell,n}$ with support $K = K_{\ell,n}$, we may take a sequence of functions $\{\psi_i\}_{i=0}^\infty$ such that $\psi_i \equiv 1$ on $\{x \in K \mid \operatorname{dist}(\partial K, x) > 1/i\}$, and $\psi_i \equiv 0$ on $\{x \in K \mid \operatorname{dist}(\partial K, x) < 2/i\}$. As smooth approximations of φ supported on K, we take we take $\varphi_i := (\rho_{1/(2i)} \circ R_{z_\Lambda}^{-1}) * (\psi_i\varphi)$, $(i = 0, 1, 2, \dots)$. Here $\{\rho_\epsilon\}_{\epsilon>0}$ are the standard mollifiers on $\mathbb{R}^{m-1} = R_{z_\Lambda}^{-1} z_\Gamma^\perp$.) □

Lemma 4.3. *Let $\mathcal{F}$ be a finite collection of maps $\psi \in C^1(\operatorname{cl}\Omega \times \mathbb{R}^m \times \mathbb{R}^m \times S^{m-1})$. Denote*

$$T_\psi u := \psi(\cdot, u^+, u^-, \nu_{J_u})\mathcal{H}^{m-1} \llcorner J_u, \quad (\psi \in \mathcal{F}). \tag{4.16}$$

Suppose that $\mathcal{F}$ includes the functions $\psi_i^\nu : (x, u^+, u^-, \nu) \mapsto \nu_i$, and $\psi_i^\pm : (x, u^+, u^-, \nu) \mapsto (u^\pm)_i$ for $i \in \{1, \dots, m\}$. Let $\{v, w^0, w^1, w^2, \dots\} \subset \mathrm{SBV}(\Omega; \mathbb{R}^K) \cap L_M^\infty(\Omega; \mathbb{R}^K)$ satisfy

$$\sup_k \mathcal{H}^{m-1}(J_{w^k}) < \infty, \tag{4.17}$$

$$\sup_k \eta(T_\psi w^k) < \infty, \quad (\psi \in \mathcal{F}), \tag{4.18}$$

$$\nu_{J_{w^k}}\mathcal{H}^{m-1}\llcorner J_{w^k} \overset{*}{\rightharpoonup} \nu_{J_v}\mathcal{H}^{m-1}\llcorner J_v \quad \textit{weakly* in } \mathcal{M}(\Omega; S^{m-1}), \quad \textit{and,} \tag{4.19}$$

$$(w^k)^\pm\mathcal{H}^{m-1}\llcorner J_{w^k} \overset{*}{\rightharpoonup} v^\pm\mathcal{H}^{m-1}\llcorner J_v \quad \textit{weakly* in } \mathcal{M}(\Omega; \mathbb{R}^m). \tag{4.20}$$

Then, after possibly moving to an unrelabelled subsequence, we have $T_\psi w^k \overset{*}{\rightharpoonup} T_\psi v$ and $|T_\psi w^k| \overset{*}{\rightharpoonup} |T_\psi v|$ for all $\psi \in \mathcal{F}$.

Proof. Let $\psi \in \mathcal{F}$. The function ψ is bounded on the compact set $\operatorname{cl}\Omega \times \operatorname{cl} B(0, M) \times \operatorname{cl} B(0, M) \times S^{m-1})$, so that, minding $\|w^k\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M$, the sequence $\{T_\psi w^k\}_{k=0}^\infty$ is also bounded in $\mathcal{M}(\Omega)$. Therefore, after possibly moving to a subsequence, we may assume the measures $\{T_\psi w^k\}_{k=0}^\infty$ to converge weakly* to some $\omega_\psi \in \mathcal{M}(\Omega)$, and the measures $\{|T_\psi w^k|\}_{k=0}^\infty$ to converge weakly* to some $\lambda_\psi \in \mathcal{M}(\Omega)$. By (4.18) and Theorem 3.1 it follows that $\lambda_\psi = |\omega_\psi|$.

The question remains, whether $\omega_\psi = T_\psi v$. Indeed, it follows from the weak* convergences (4.19) and (4.20) that $\omega_\psi = T_\psi v$ for $\psi = \psi_i^\nu, \psi_i^\pm$, $(i = 1, \dots, m)$. In particular

$$\mu_{w^k} \overset{*}{\rightharpoonup} \mu_v \quad \text{and} \quad |\mu_{w^k}|(\Omega) \to |\mu_v|(\Omega). \tag{4.21}$$

for $\mu_u := (u^+, u^-, \nu_{J_u})\mathcal{H}^{m-1}\llcorner J_u \in \mathcal{M}(\Omega; \mathbb{R}^m \times \mathbb{R}^m \times S^{m-1})$.

Minding that $\|\nu_{J_u}(x)\| = 1$, we may now write for $f \in C_c^\infty(\Omega)$ and

$$\psi_f(x, a, b, z) := f(x)\psi\left(x, \frac{a}{\|z\|}, \frac{b}{\|z\|}, \frac{z}{\|z\|}\right)\|z\|$$

that

$$\begin{aligned}
\int_\Omega f(x)\, dT_\psi u(x) &= \int_\Omega f(x)\psi\big(x, u^+(x), u^-(x), \nu_{J_u}(x)\big)\, d\mathcal{H}^{m-1}\llcorner J_u \\
&= \int_\Omega f(x)\frac{\psi\big(x, u^+(x), u^-(x), \nu_{J_u}(x)\big)}{\big\|\big(u^+(x), u^-(x), \nu_{J_u}(x)\big)\big\|}\, d|\mu_u|(x). \\
&=: \int_\Omega \psi_f\left(x, \frac{d\mu^k}{d|\mu^k|}\right) d|\mu_u|(x).
\end{aligned}$$

The function ψ_f is continuous, because ψ is C^1, $\|\nu_{J_u}(x)\| = 1$, and

$$1/\|z(x)\| = \|(u^+(x), u^-(x), \nu_{J_u}(x))\|/\|\nu_{J_u}(x)\| = \|(u^+(x), u^-(x), \nu_{J_u}(x))\| \leq \sqrt{2M^2+1}.$$

It therefore follows from the Reshetnyak continuity theorem (see, e.g., [3, Theorem 2.39]) and (4.21) that $T_\psi w^k \overset{*}{\rightharpoonup} T_\psi v$. Hence $\mu_\psi = T_\psi v$. □

Next we prove a trace result.

Proposition 4.1. *Let $V \subset \mathbb{R}^{m-1}$ be an open and bounded, $f : V \to \mathbb{R}$ Lipschitz continuous of factor L, and $\varrho > 0$. Define*

$$\Omega := \{(x, s) \in V \times \mathbb{R} \mid s \in f(x) + (-\varrho, \varrho)\},$$

and $g(x) := (x, f(x))$. Suppose $u \in W^{1,\infty}(\Omega)$. Then u has a trace u_Γ on $\Gamma := g(V)$, and $u_\Gamma \circ g \in W^{1,\infty}(V)$ with

$$\|u_\Gamma \circ g\|_{W^{1,\infty}(V)} \leq C_{15}\|u\|_{W^{1,\infty}(\Omega)} \tag{4.22}$$

for some constant $C_{15} = C_{15}(L, m)$.

Proof. The existence of a trace $u_\Gamma \in L^1(\Gamma)$ follows from standard results. We just have show that $u_\Gamma \circ g$ is Lipschitz on V. Let us set $U := V \times (-\varrho, \varrho)$ and

$$v(x, s) := u(x, f(x) + s) = u(\widetilde{g}(x, s)) \quad ((x, s) \in U),$$

where $\widetilde{g}(x, s) := g(x) + (0, s)$. We have

$$\nabla\widetilde{g}(x, s) = \begin{pmatrix} \nabla g(x) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad ((x, s) \in U),$$

as well as

$$\nabla v(x, s) = \nabla\widetilde{g}(x, s)\nabla u(\widetilde{g}(x, s)),$$

so that clearly $v \in W^{1,\infty}(U)$ with the bound

$$\|v\|_{W^{1,\infty}(U)} \leq C_{16}\|u\|_{W^{1,\infty}(\Omega)} \tag{4.23}$$

for some constant $C_{16} = C_{16}(L, m)$.

Since u is (Lipschitz) continuous, as is v, we observe that $u_\Gamma \circ g = v_0 := v(\cdot, 0)$. But clearly, still by continuity, Lipschitz continuity is preserved by traces on affine sets, in particular on $V \times \{0\}$. We therefore obtain

$$\|v_0\|_{W^{1,\infty}(V)} \leq \|v\|_{W^{1,\infty}(U)}. \tag{4.24}$$

Combining (4.23), (4.24) shows (4.22). □

Proposition 4.2. *Let $V \subset \mathbb{R}^{m-1}$ be an open and bounded, $f : V \to \mathbb{R}$ Lipschitz continuous of factor L, and $\varrho > 0$. Define*

$$\Omega := \{(x, s) \in V \times \mathbb{R} \mid s \in f(x) + (-\varrho, \varrho)\}, \quad \Omega^\pm := \{(x, s) \in V \times \mathbb{R} \mid s \in f(x) + (0, \pm\varrho)\},$$

and $g(x) := (x, f(x))$. Let $\Gamma := g(V)$. Suppose $u \in W^{1,\infty}(\Omega\setminus\Gamma)$ with $\mathcal{H}^{m-1}(\{x \in \Gamma \mid u^+(x) - u^-(x)\}) = 0$. Then there exist extensions $v^{(\pm)} \in W^{1,\infty}(\Omega)$ of $u|\Omega^\pm$, satisfying

$$\|v^{(\pm)}\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega^\pm)} \quad \textit{and} \quad \|v^{(\pm)}\|_{W^{1,\infty}(\Omega)} \leq C_{17}\|u\|_{W^{1,\infty}(\Omega^\pm)} \tag{4.25}$$

for some $C_{17} = C_{17}(L, m, u)$. Moreover

$$\mathcal{L}^m(\{x \in \Omega \mid v^{(+)}(x) = v^{(-)}(x)\} = 0. \tag{4.26}$$

Proof. From Proposition 4.1, we deduce that

$$\|u^{\pm} \circ g\|_{W^{1,\infty}(V)} \leq C_{15}\|u\|_{W^{1,\infty}(\Omega)}$$

for $C_{15} = C_{15}(L, m)$. Let $q_0, q_1 : \mathbb{R}^+ \to \mathbb{R}^+$ be the saw-tooth functions that oscillate between the values 0 and 1 at slope $|q_0'| = |q_1'| = 2\|\nabla u\|_{L^\infty(\Omega)}$, with initial values $q_0(0) = 0$ and $q_1(0) = 1$. Let $p(x) := g(P^{\perp}_{(0,1)}(x))$ be the projection of x on Γ (along $z_\Gamma = (1, 0)$). Then the functions $u^{\pm} \circ p$ are Lipschitz with factor at most $L\|\nabla u\|_{L^\infty(\Omega^{\pm};\mathbb{R}^m)}$. Consequently, defining

$$v^{(\pm)}(x) = \begin{cases} u(x), & x \in \Omega^{\pm}, \\ q_1(\|x - p(x)\|)u^{\pm}(p(x)) + q_0(\|x - p(x)\|)u^{\mp}(p(x)), & x \in \Omega^{\mp}, \end{cases}$$

and minding that $u^{\pm}$ and q_0, q_1 are bounded, we find that $v^{\pm}$ are Lipschitz and (4.25) holds for some $C_{17} = C_{17}(L, m, u)$. Moreover, we deduce (4.26) thanks to $\mathcal{H}^{m-1}(\{x \in \Gamma \mid u^+(x) - u^-(x)\}) = 0$ and

$$\mathcal{L}^1(\{s \in f(x) + (-\delta, \delta) \mid v^{(+)}(x, s) = v^{(-)}(x, s)\}), \quad \text{(a.e. } x \in V).$$

The latter follows from the fact that by construction the functions $x \mapsto q_i(\|x - p(x)\|)$, $(i = 0, 1,)$, oscillate faster than u on lines $\{y\} \times \mathbb{R}$, $(y \in V)$. □

Remark 4.1. The property (4.26) together with preserving the L^∞ bound in (4.25) are the reason for not using standard Sobolev or Lipschitz (cf. [10]) extension results.

Remark 4.2. Both Proposition 4.1 and Proposition 4.2 can easily by a rotation argument be extended to domains $\Omega = g_\Gamma(V_\Gamma) + z_\Gamma(-\varrho, \varrho)$ defined by a general Lipschitz graph Γ.

5. The space and boundary covers

We now introduce the space $\mathcal{A}(\Omega; \mathbb{R}^K)$ of functions admissible for the approximation theorem stated in the next section.

Definition 5.1. Given an open set $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary, we denote by $\mathcal{A}(\Omega; \mathbb{R}^K)$ the set of functions $u : \Omega \to \mathbb{R}^K$ that are in $W^{1,\infty}(\Omega \setminus J; \mathbb{R}^K)$ for a (with respect to Ω) compact set $J = \widehat{J}_u \subset \Omega$ satisfying the following:

(i) $\mathcal{H}^{m-1}(J \setminus J_u) = 0$.
(ii) $J = \bigcup_{i=1}^N \Lambda_i$, where Λ_i is a Lipschitz $(m-1)$-graph of constant at most L.
(iii) $\Lambda_i \cap \Lambda_n \subset \partial\Lambda_i \cup \partial\Lambda_n$ and $\Lambda_i \cap \partial\Omega \subset \partial\Lambda_i$. with $\partial\Lambda_i := g_{\Lambda_i}(\partial V_{\Lambda_i})$, $(i, n = 1, \ldots, N;\ i \neq n)$,
(iv) $J \in \mathrm{Sp}^{m-1}(\Omega, L, M)$ for some $M \in (0, \infty)$.
(v) Each V_{Λ_i}, $(i = 1, \ldots, N)$ has Lipschitz boundary.
(vi) $\nabla g_{\Lambda_i} \in \mathrm{BV}(V_{\Lambda_i}; \mathbb{R}^m \times z^{\perp}_{\Lambda_i})$, $(i = 1, \ldots, N)$.

We will henceforth use the shorthand notation $V_i := V_{\Lambda_i}$, $g_i := g_{\Lambda_i}$, and $z_i := z_{\Lambda_i}$.

Remark 5.1. Observe that if $\{u^i\}_{i=0}^\infty \subset \mathcal{A}(\Omega; \mathbb{R}^K)$ with the same constants L, M, i.e., $\widehat{J}_{u^i} \in \mathrm{Sp}^{m-1}(\Omega, L, M)$, and if

$$\sup_i \|u^i\|_{W^{1,\infty}(\Omega \setminus \widehat{J}_{u^i})} + \mathcal{H}^{m-1}(\widehat{J}_{u^i}) + \mathrm{Sp}(\widehat{J}_{u^i}) < \infty,$$

then it follows from Theorem 3.2 and Proposition 3.2 that there exists $u \in \mathrm{SBV}(\Omega; \mathbb{R}^K)$ with $\widehat{J}_u \in \mathrm{Sp}^{m-1}(\Omega, L, M)$ such that the convergences (3.24)–(3.27) hold for a subsequence. Similar closure properties for sets within the space $\mathcal{A}(\Omega; \mathbb{R}^K)$ itself would depend on further limiting the complexity and number of the graphs $\{\Lambda_i\}_{i=1}^N$.

In the remainder of this section we provide a series of technical lemmas studying the covering of $\bigcup_{i=1}^N \partial\Lambda_i$ by cubes on a grid. We begin by definitions related to the cover.

Definition 5.2. We denote $rQ := [0,r]^m$ and $rQ_0 := [0,r)^m$ for $r > 0$.

Definition 5.3. Suppose $Z = X + rQ$ for some set $X \subset y + r\mathbb{Z}^m$ with $r > 0$ and $y \in Q_0$. We then say that $E \subset \partial Z$ is a *face* of Z if for some $\xi \in X$ the set $E - \xi$ is a face of rQ, i.e., for some $i = 1, \ldots, m$ and $\theta \in \{0,1\}$, we have $E = \xi + r\{x \in Q \mid \langle x, e_i\rangle = \theta\}$.

Definition 5.4. Suppose $J = \bigcup_{i=1}^N \Lambda_i$ is as in Definition 5.1. Denote $\widetilde{\partial} J := \bigcup_{i=1}^N \partial\Lambda_i$. Then for $r > 0$ and $y \in Q_0$, we let

$$\begin{aligned} \bar{F}_r &:= \{\xi \in r\mathbb{Z}^m \mid (\xi + 2rQ) \cap \widetilde{\partial} J \neq \emptyset\}, \\ F_r^y &:= ry + \bar{F}_r, \quad \text{and} \\ Z_r^y &:= F_r^y + rQ. \end{aligned}$$

The sets Z_r^y, $(y \in Q_0)$, are the covers of the boundary we are interested in. We now show a bound on the size of the cover, and then an average density estimate for sets in the neighbourhood of this family of covers. Then we will prove further lemmas.

Lemma 5.1. *Let J be as in Definition 5.1. There then exists a constant $C_{18} = C_{18}(J)$ such that for each $r > 0$ and $i = 1, \ldots, N$ there are $K \leq Cr^{2-m}$ open balls $B_1, \ldots, B_K$ of diameter at most r with $\partial V_{\Lambda_i} \subset \bigcup_{k=1}^K B_k$.*

Proof. This is a consequence of the Lipschitz boundary property Definition 5.1(v). We take an open cover $U_1, \ldots, U_M$ of ∂V_{Λ_i} such that $\partial V_{\Lambda_i} \cap U_n$ is a Lipschitz graph (in the $(m-1)$-dimensional space $z_{\Lambda_i}^\perp$) for each $n = 1, \ldots, M$. Each of the sets $\partial V_{\Lambda_i} \cap U_n$, may, as a Lipschitz graph of dimension $m-2$, trivially be covered by $C_{i,n} r^{2-m}$ open balls of diameter at most r, for some $C_{i,n} = C_{i,n}(J)$. □

Lemma 5.2. *$\#\bar{F}_r \leq C_{19} r^{2-m}$ for $C_{19} = C_{19}(J)$.*

Proof. One simply considers the cover of ∂V_i by $K \leq Cr^{2-m}$ balls $B_1, \ldots B_K$ of diameter r from Lemma 5.1. Since g_i is Lipschitz of factor at most L, covering the images $g_i(B_n)$ by squares $rQ + \xi$ with $\xi \in r\mathbb{Z}^m$ produces the claim. □

Lemma 5.3. *Let J be as in Definition 5.1 and $J' = \bigcup_{i=1}^{N'} \Lambda_i'$ for Lipschitz $(m-1)$-graphs $\{\Lambda_i'\}_{i=1}^{N'}$. Then there exists a constant $C_{20} = C_{20}(J, N', m)$ such that for every $r > 0$ and $h \in (0, r]$, we have the bound*

$$\int_{Q_0} \mathcal{H}^{m-1}\big(J' \cap (Z_r^y + B(0,h)) \setminus Z_r^y\big)\, dy \leq C_{20} h. \tag{5.1}$$

Proof. As $\chi_{F_r^y + rQ}(x) = \sum_{\xi \in \bar{F}_r} \chi_{\xi + ry + rQ}(x)$ for $\mathcal{L}^m$-a.e. $y \in Q_0$, we begin by calculating

$$\int_{Q_0} \chi_{F_r^y + rQ}(x)\, dy = \int_{Q_0} \sum_{\xi \in \bar{F}_r} \chi_{\xi + ry + rQ}(x)\, dy = r^{-m} \sum_{\xi \in \bar{F}_r} \int_{rQ_0} \chi_{\xi + y + rQ}(x)\, dy.$$

Using $\chi_{F_r^y + rQ + B(0,h)}(x) \leq \sum_{\xi \in \bar{F}_r} \chi_{\xi + ry + rQ + B(0,h)}(x)$, we similarly get the inequality

$$\int_{Q_0} \chi_{F_r^y + rQ + B(0,h)}(x)\, dy \leq r^{-m} \sum_{\xi \in \bar{F}_r} \int_{rQ_0} \chi_{\xi + y + rQ + B(0,h)}(x)\, dy.$$

Denoting the left hand side of (5.1) by $A_{r,h}$, we may now write

$$\begin{aligned}
A_{r,h} &= \int_{Q_0}\int_{J'} \chi_{Z_r^y+B(0,y)}(x) - \chi_{Z_r^y}(x)\, d\mathcal{H}^{m-1}(x)\, dy \\
&= \int_{J'}\int_{Q_0} \chi_{F_r^y+rQ+B(0,h)}(x) - \chi_{F_r^y+rQ}(x)\, dy\, d\mathcal{H}^{m-1}(x) \\
&\leq r^{-m}\int_{J'}\sum_{\xi\in\bar{F}_r}\int_{rQ_0} \chi_{\xi+y+rQ+B(0,h)}(x) - \chi_{\xi+y+rQ}(x)\, dy\, d\mathcal{H}^{m-1}(x) \\
&= r^{-m}\sum_{\xi\in\bar{F}_r}\int_{J'}\int_{rQ_0} \chi_{(\xi+rQ+B(0,h))\setminus(\xi+rQ)}(x-y)\, dy\, d\mathcal{H}^{m-1}(x).
\end{aligned}$$

Employing the fact that $J' = \bigcup_{i=1}^{N'}\Lambda_i'$ with Λ_i' (Lipschitz) graphs, we deduce the existence of a constant $C_{21} = C_{21}(N', m)$ such that

$$\int_{J'}\int_{rQ_0} \chi_E(x-y)\, dy\, d\mathcal{H}^{m-1}(x) \leq C_{21}r^{m-1}\int_{J'-B(0,rm)} \chi_E(x)\, dx \tag{5.2}$$

for Borel sets E. Indeed, let $\Lambda = \Lambda_i'$ and $z = z_{\Lambda_i'}$. Then, since

$$Q_0 \subset P_zQ_0 + P_z^{\perp}Q_0 \subset B(0,m),$$

we have

$$\begin{aligned}
\int_{\Lambda}\int_{rQ_0} \chi_E(x-y)\, dy\, d\mathcal{H}^{m-1}(x) &\leq \int_{\Lambda}\int_{P_zrQ_0}\int_{P_z^{\perp}rQ_0} \chi_E((x-t)-y)\, dy\, dt\, d\mathcal{H}^{m-1}(x) \\
&= \int_{P_z^{\perp}rQ_0}\int_{\Lambda-P_zrQ_0} \chi_E(x-y)\, dx\, dy \\
&\leq \int_{P_z^{\perp}rQ_0} dy \int_{\Lambda-P_zrQ_0-P_z^{\perp}rQ_0} \chi_E(x)\, dx \\
&\leq C_{22}r^{m-1}\int_{\Lambda-B(0,rm)} \chi_E(x)\, dx.
\end{aligned}$$

In the final step we have employed the fact that $\mathcal{L}^{m-1}(P_z^{\perp}rQ_0) \leq C_{22}r^{m-1}$ for some constant $C_{22} = C_{22}(m)$. Summing over the estimates for $\Lambda = \Lambda_1', \ldots, \Lambda_{N'}'$ now gives (5.2).

With (5.2) at our disposal, we may now calculate that

$$\begin{aligned}
A_{r,h} &\leq C_{21}r^{-m}\sum_{\xi\in\bar{F}_r} r^{m-1}\int_{J'-B(0,rm)} \chi_{(\xi+rQ+B(0,h))\setminus(\xi+rQ)}(x)\, dx \\
&= C_{21}r^{-1}\sum_{\xi\in\bar{F}_r}\mathcal{L}^m\big((J'-B(0,rm))\cap(\xi+rQ+B(0,h))\setminus(\xi+rQ)\big) \\
&\leq C_{21}r^{-1}\sum_{\xi\in\bar{F}_r}\mathcal{L}^m\big((\xi+rQ+B(0,h))\setminus(\xi+rQ)\big) \\
&\leq C_{21}C_{23}hr^{m-2}\#\bar{F}_r
\end{aligned} \tag{5.3}$$

Here we have finally employed the assumption $h \in (0, r]$, from which it follows that

$$\mathcal{L}^m\big((rQ+B(0,h))\setminus rQ\big) \leq C_{23}hr^{m-1}$$

for some $C_{23} = C_{23}(m)$. By Lemma 5.2, we have $\#\bar{F}_r \leq C_{19}r^{2-m}$. Hence

$$A_{r,h} \leq C_{21}C_{23}C_{19}h,$$

which gives (5.1). □

Lemma 5.4. *Let J be as in Definition 5.1 and $J' = \bigcup_{i=1}^{N'} \Lambda'_i$ for Lipschitz $(m-1)$-graphs $\{\Lambda'_i\}_{i=1}^{N'}$. Then*

$$\int_{Q_0} \mathcal{H}^{m-2}(J' \cap \partial Z_r^y)\, dy \le C_{24}, \quad (r > 0),$$

for some $C_{24} = C_{24}(J, N', m)$.

Proof. Let $H_r := \sum_{i=1}^m (r\mathbb{Z}e_i + e_i^\perp)$. We observe that

$$J' \cap \partial Z_r^y \subset J' \cap Z_r^y \cap (ry + H_r) \subset \big(J' \cap \bigcup_{y' \in Q_0} Z_r^{y'}\big) \cap (ry + H_r).$$

Pick any $\widehat{y} \in Q_0$. Then

$$\bigcup_{y' \in Q_0} Z_r^{y'} = \bigcup_{y' \in Q_0} (\bar{F}_r + ry' + rQ) \subset \bar{F}_r + r\widehat{y} + rQ + [-1,1]rQ = Z_r^{\widehat{y}} + [-1,1]rQ,$$

so that setting

$$J_r^{\widehat{y}} := J' \cap (Z_r^{\widehat{y}} + [-1,1]rQ),$$

gives

$$J' \cap \partial Z_r^y \subset J_r^{\widehat{y}} \cap (ry + H_r). \tag{5.4}$$

Next we deduce for some $C_{25} = C_{25}(J, N', m)$ that

$$\int_{Q_0} \mathcal{H}^{m-1}(J' \cap (Z_r^y + [-1,1]rQ)) \le C_{25} r.$$

This can be shown analogously to Lemma 5.3, minding in the step corresponding to (5.3) that

$$\mathcal{L}^m((J' - B(0, rm)) \cap (\xi + rQ + [-1,1]rQ)) \le (3r)^m.$$

We may therefore choose $\widehat{y} \in Q_0$ with

$$\mathcal{H}^{m-1}(J_r^{\widehat{y}}) = \mathcal{H}^{m-1}(J' \cap (Z_r^{\widehat{y}} + [-1,1]rQ)) \le C_{25} r.$$

The claim of the present lemma is now established by reasoning

$$\begin{aligned}
\int_{Q_0} \mathcal{H}^{m-2}(J' \cap \partial Z_r^y)\, dy &\le \int_{Q_0} \mathcal{H}^{m-2}(J_r^{\widehat{y}} \cap (ry + H_r))\, dy \\
&\le \sum_{i=1}^m \int_{Q_0} \mathcal{H}^{m-2}(J_r^{\widehat{y}} \cap (ry + r\mathbb{Z}e_i + e_i^\perp))\, dy \\
&= \sum_{i=1}^m \sum_{n \in \mathbb{Z}} \int_0^1 \mathcal{H}^{m-2}(J_r^{\widehat{y}} \cap (r(s+n)e_i + e_i^\perp))\, ds \\
&= \sum_{i=1}^m \int_{\mathbb{R}} \mathcal{H}^{m-2}(J_r^{\widehat{y}} \cap (rse_i + e_i^\perp))\, ds \\
&\le \frac{m}{r} \mathcal{H}^{m-1}(J_r^{\widehat{y}}) \le C_{25} m.
\end{aligned}$$

In the first inequality we have employed (5.4), and in the second-to-last inequality the coarea formula. □

Lemma 5.5. *Let $J = \bigcup_{i=1}^N \Lambda_i$ be as in Definition 5.1. Then $\widetilde{\partial} J \subset \operatorname{int} \bigcap_{y \in Q_0} Z_r^y$.*

Proof. First we observe that

$$\widetilde{\partial}J \subset \operatorname{int}\big((\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ\big). \tag{5.5}$$

Indeed, let $x = (x_1, \dots, x_m) \in \widetilde{\partial}J$. For any $i \in \{1, \dots, m\}$, if there exists $z \in (x_i - (0, r)) \cap r\mathbb{Z}$, then clearly

$$x_i \in \operatorname{int}\big(z + [0, r]\big) \subset \operatorname{int}\Big(\big[(\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ\big]_i\Big).$$

Otherwise, if $(x_i - (0, r)) \cap r\mathbb{Z} = \emptyset$, then $x_i \in r\mathbb{Z}$. It follows that

$$x_i \in \operatorname{int}\big((x_i - r + [0, r]) \cup (x_i + [0, r])\big) \subset \operatorname{int}\Big(\big[(\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ\big]_i\Big).$$

We conclude that (5.5) holds.

Next we observe that

$$(\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ \subset (\widetilde{\partial}J - 2rQ) \cap r\mathbb{Z}^m + ry + rQ = Z_r^y, \quad (y \in Q_0). \tag{5.6}$$

Indeed, let again $x = (x_1, \dots, x_m)$ satisfy $x \in (\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ$. Then

$$x_i = rk + ra \quad \text{and} \quad rk = z - rq$$

for some $k \in \mathbb{Z}$, $a \in [0, 1]$, $z \in \widetilde{\partial}J$ and $q \in [0, 1]$. We want to show that

$$x_i = rn + ry + rb \quad \text{and} \quad rn = \bar{z} - 2rp$$

for some $b \in [0, 1]$, $n \in \mathbb{Z}$, $\bar{z} \in \widetilde{\partial}J$ and $p \in [0, 1]$.

If $a \geq y$, this is satisfied when $b = a - y$ and $n = k$, as well as $\bar{z} = z$ and $p = q$.

If $a < y$, we pick $b = 1 - y + a$ and $n = k - 1$, as well as $p = (q + 1)/2$ and $\bar{z} = z$.

We have thus shown (5.6), whence also

$$(\widetilde{\partial}J - rQ) \cap r\mathbb{Z}^m + rQ \subset \bigcap_{y \in Q_0} Z_r^y.$$

Recalling (5.5) it now follows that $\widetilde{\partial}J \subset \operatorname{int} \bigcap_{y \in Q_0} Z_r^y$. □

Lemma 5.6. *Let $J = \bigcup_{i=1}^N \Lambda_i$ be as in Definition 5.1 and J' a $\mathcal{H}^{m-1}$-rectifiable set. Pick $r > 0$, some $y_r \in Q_0$, as well as ℓ satisfying $h_\ell \in (0, r)$, Define $Z_r := Z_r^{y_r}$, $F_r := F_r^{y_r}$, and*

$$\mu_{r,\ell} := \mathcal{H}^{m-1} \llcorner \partial Z_r + \mathcal{H}^{m-1} \llcorner (J' \setminus Z_r).$$

Then

$$\mathrm{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_\ell) \leq \mathcal{H}^{m-1}(J' \setminus Z_r) + C_{28} h_\ell \tag{5.7}$$

for some $C_{28} = C_{28}(J)$ and

$$\mathcal{G}_\ell := \{\Gamma_\ell^x := \partial Z_r \cap B(x, h_\ell) \mid B(x, h_\ell) \text{ intersects at most one face of } Z_r\}.$$

Proof. Denote by $E_{r,\ell}$, $(\ell = 0, 1, 2, \dots)$ the points $x \in \mathbb{R}^m$ such that $B(x, h_\ell)$ touches more than one face of Z_r. Then $B(x, \sqrt{m} h_\ell)$ touches more than one face of some cube $\xi + rQ$, $\xi \in F_r$. Consequently,

$$E_{r,\ell} \subset F_r + rH + B(x, \sqrt{m} h_\ell),$$

where H denotes the union of all the edges of Q, of the form

$$\{z \in Q \mid \langle e_i, z \rangle = \theta_i, \langle e_k, z \rangle = \theta_k\}, \quad \text{where} \quad i, k = 1, \dots, m;\ i \neq k;\ \theta_i \in \{0, 1\}.$$

We may now calculate that

$$\begin{aligned}\int_{E_{r,\ell}} (\mathcal{H}^{m-1} \llcorner \partial Z_r)(\tau_x f_\ell)\, dx &\leq (\mathcal{H}^{m-1} \llcorner \partial Z_r)(E_{r,\ell} + B(0,h_\ell)) \\ &\leq \sum_{\xi \in F_r} (\mathcal{H}^{m-1} \llcorner \partial Z_r)(\xi + rH + B(0,(1+\sqrt{m})h_\ell)) \\ &\leq \# F_r C_{26} \mathcal{H}^{m-1}(r\partial Q \cap (rH + B(0, 2\sqrt{m}h_\ell)))\end{aligned}$$

for some $C_{26} = C_{26}(m)$. We recall that $\# F_r^{\leq} C_{19} r^{2-m}$. If $2\sqrt{m}h_\ell < r$, we may thus continue to calculate

$$\# F_r C_{26} \mathcal{H}^{m-1}(r\partial Q \cap (rH + B(0, 2\sqrt{m}h_\ell))) \leq \# F_r C_{27} r^{m-2} h_\ell \leq C_{28} h_\ell$$

for some constants $C_{27} = C_{27}(m)$ and $C_{28} = C_{28}(J,m)$. If, on the other hand, $2\sqrt{m}h_\ell \geq r$, we may calculate

$$\# F_r C_{26} \mathcal{H}^{m-1}(r\partial Q \cap (rH + B(0, 2\sqrt{m}h_\ell))) \leq \# F_r C_{27} r^{m-1} = C_{19} C_{27} r \leq C_{28} h_\ell.$$

Thus

$$\int_{E_{r,\ell}} (\mathcal{H}^{m-1} \llcorner \partial Z_r)(\tau_x f_\ell)\, dx \leq C_{28} h_\ell. \tag{5.8}$$

Minding the definition of $\mu_{r,\ell}$, and recalling from Definition 3.2 the notation $O_\ell^x := \operatorname{supp} \tau_x f_\ell$, we can continue to calculate

$$\begin{aligned}\int_{E_{r,\ell}} |\mu_{r,\ell}|(\tau_x f_\ell)\, dx &\leq \int_{E_{r,\ell}} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \partial Z_r|(\tau_x f_\ell)\, dx + \int_{E_{r,\ell}} (\mathcal{H}^{m-1} \llcorner \partial Z_r)(\tau_x f_\ell)\, dx \\ &\leq \int_{E_{r,\ell}} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \partial Z_r|(\tau_x f_\ell)\, dx + C_{28} h_\ell.\end{aligned} \tag{5.9}$$

Let us then observe that, by the choice of Γ_ℓ^x, since $B(x, h_\ell)$ for $x \in \mathbb{R}^m \setminus E_{r,\ell}$ intersects at most one face of ∂Z_r, we have

$$\int_{\mathbb{R}^m \setminus E_{r,\ell}} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx = \int_{\mathbb{R}^m \setminus E_{r,\ell}} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \partial Z_r|(\tau_x f_\ell)\, dx,$$

so that combining with (5.9) yields

$$\begin{aligned}\operatorname{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_\ell) &= \int_{E_{r,\ell}} |\mu_{r,\ell}|(\tau_x f_\ell)\, dx + \int_{\mathbb{R}^m \setminus E_{r,\ell}} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx \\ &\leq \int_{\mathbb{R}^m} |\mu_{r,\ell} \llcorner O_\ell^x \setminus \partial Z_r|(\tau_x f_\ell)\, dx + C_{28} h_\ell.\end{aligned} \tag{5.10}$$

Minding the definition of $\mu_{r,\ell}$, we get

$$|\mu_{r,\ell} \llcorner O_\ell^x \setminus \partial Z_r|(\tau_x f_\ell) = (\mathcal{H}^{m-1} \llcorner J' \setminus Z_r)(\tau_x f_\ell).$$

Thus (5.7) follows from (5.10). □

Remark 5.2. Each $\Gamma_\ell^x \in \mathcal{G}_\ell$ in the above lemma is clearly a Lipschitz graph that satisfies (3.13).

6. The main approximation theorem

We now reach our main result. The space $\mathcal{A}(\Omega; \mathbb{R}^K)$ of admissible functions is defined in Definition 5.1, and the operators T_ψ, $(\psi \in \mathcal{F})$ in (4.16). We recall that the same (fixed) regular nested sequence of functions $\{(f_\ell, \nu_\ell)\}_{\ell=0}^\infty$ with corresponding regularity constants $\{h_\ell\}_{\ell=0}^\infty$ (see Definition 3.1) is used for the definition of both η and Sp (see Theorem 3.1 and Definition 3.2, respectively).

Theorem 6.1. *Suppose $u \in \mathcal{A}(\Omega;\mathbb{R}^K)$. Let $\mathcal{F}$ be a finite collection of maps $\psi \in C^1(\operatorname{cl}\Omega \times \mathbb{R}^m \times \mathbb{R}^m \times S^{m-1})$. Then there exists a sequence $\{u^i\}_{i=0}^\infty \subset \mathcal{A}(\Omega;\mathbb{R}^K)$ such that each set $\widehat{J}_{u^i}$ from Definition 5.1 is polyhedral, and*

$$u^i \to u \text{ strongly in } L^2(\Omega;\mathbb{R}^m), \tag{6.1}$$

$$\nabla u^i \to \nabla u \text{ strongly in } L^2(\Omega;\mathbb{R}^{K\times m}), \tag{6.2}$$

$$D^j u^i \overset{*}{\rightharpoonup} D^j u \text{ weakly* in } \mathcal{M}(\Omega;\mathbb{R}^{K\times m}), \tag{6.3}$$

$$\mathcal{H}^{m-1}(J_{u^i}) \to \mathcal{H}^{m-1}(J_u), \tag{6.4}$$

$$T_\psi u^i \overset{*}{\rightharpoonup} T_\psi u \text{ weakly* in } \mathcal{M}(\Omega), \quad \text{and} \tag{6.5}$$

$$\eta(T_\psi u^i) \to \eta(T_\psi u), \quad (\psi \in \mathcal{F}). \tag{6.6}$$

In particular, it can be ensured that $|D^j u^i|(\Omega) \to |D^j u|(\Omega)$ and $|\operatorname{Div}^j u^i|(\Omega) \to |\operatorname{Div}^j u|(\Omega)$.

Proof. We divide the proof into three steps: (Step 1) Construction of approximating sequences, (Step 2) convergence of the preliminary approximations v_r to u, and (Step 3) convergence of the approximations w_r^k to the preliminary approximations v_r.

Step 1: Construction of approximating sequences We let $\{\Lambda_i\}_{i=1}^N$ be the Lipschitz graphs from Definition 5.1 for u and use the shorthand notation $J = \widehat{J}_u$. We let $M_u := \|u\|_{L^\infty(\Omega;\mathbb{R}^K)}$ and denote by L the maximal Lipschitz factor of $g_i := g_{\Lambda_i}$, $(i = 1,\dots,N)$. We pick $r \in (0,1)$, fixed for the moment. We recall from Definition 5.4 that

$$\begin{aligned}
\widetilde{\partial} J &:= \bigcup_{i=1}^N \partial\Lambda_i, \\
\bar{F}_r &:= \{\xi \in r\mathbb{Z}^m \mid (\xi + 2rQ) \cap \widetilde{\partial} J \neq \emptyset\}, \\
F_r^y &:= ry + \bar{F}_r, \quad \text{and} \\
Z_r^y &:= F_r^y + rQ.
\end{aligned}$$

We further let

$$\widetilde{Z}_r := \bigcap_{y \in Q_0} Z_r^y.$$

Definition 5.1(iii) and Lemma 5.5 then yield that

$$\Lambda_i \cap \Lambda_n \subset \widetilde{\partial} J \subset \operatorname{int}\widetilde{Z}_r \quad \text{and} \quad \Lambda_i \cap \partial\Omega \subset \operatorname{int}\widetilde{Z}_r, \quad (i \neq n), \tag{6.7}$$

With $\bar{s}_r \in (0,r)$ still to be determined, let us set (see Figure 2)

$$\begin{aligned}
\widehat{Z}_r &:= \{x \in \widetilde{Z}_r \mid \min_{x' \in \partial\widetilde{Z}_r} \|x - x'\| \geq \bar{s}_r\}, \quad \text{and} \\
U_{i,r} &:= (\Lambda_i \setminus \widehat{Z}_r) + (-1,1)\bar{s}_r z_i, \quad (i = 1,\dots,N),
\end{aligned}$$

and denote by $U_{i,r}^\pm$ the halves into which $U_{i,r}$ split by Λ_i. From the fact that $\Lambda_i \cap \partial\Omega \subset \partial\Lambda_i$ (Definition 5.1(iii)), we deduce that $U_{i,r} \subset \Omega$ for small enough $\bar{s}_r$. Moreover, we may and do choose $\bar{s}_r$ such that

$$\begin{gathered}
\mathcal{H}^{m-1}(\partial\widehat{Z}_r \cap J) = 0, \quad (\text{as we can pick } \mathcal{H}^{m-2}(\partial\widehat{Z}_r \cap J) < \infty), \\
\Lambda_i \cap \Lambda_n \subset \operatorname{int}\widehat{Z}_r, \quad (i \neq n), \quad (\text{minding (6.7)}), \\
\partial U_{i,r} \setminus (\Lambda_i + \{-1,1\}\bar{s}_r z_i) \subset \widetilde{Z}_r, \\
\partial\Lambda_i \cap U_{i,r} = \emptyset \quad \text{and} \\
U_{i,r} \cap (\Lambda_n \cup U_{n,r}) = \emptyset, \quad (i \neq n).
\end{gathered} \tag{6.8}$$

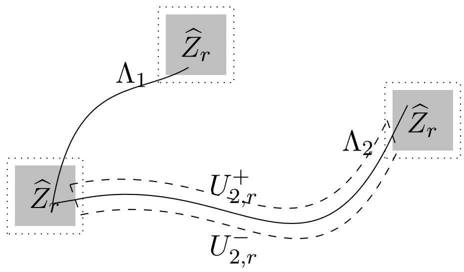

Figure 2: Some of the construction in Theorem 6.1. The dotted line represents $\widetilde{Z}_r \supset \widehat{Z}_r$. The dashed line bounds $U_{2,r}$ and together with Λ_2, the sides $U^\pm_{2,r}$.

Next, we approximate the surfaces $\Lambda_i \setminus \widehat{Z}_r$. We choose a sequence $\{s_r^k\}_{k=0}^\infty \subset (0, \bar{s}_r)$ with $s_r^k \searrow 0$. Lemma 4.2 then gives sequences $\{\Lambda^k_{i,r}\}_{k=0}^\infty$, $(i = 1, \dots, N)$, of polyhedral Lipschitz graphs of factor at most L', satisfying

$$\mathcal{H}^{m-1} \llcorner \Lambda^k_{i,r} \stackrel{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner \Lambda_i \setminus \widehat{Z}_r \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m), \tag{6.9}$$

$$\nu_{\Lambda^k_{i,r}} \mathcal{H}^{m-1} \llcorner \Lambda^k_{i,r} \stackrel{*}{\rightharpoonup} \nu_{\Lambda_i} \mathcal{H}^{m-1} \llcorner \Lambda_i \setminus \widehat{Z}_r \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m; S^{m-1}), \tag{6.10}$$

$$\Lambda^k_{i,r} \subset \Lambda_i \setminus \widehat{Z}_r + B(0, s_r^k/2), \quad \text{and} \tag{6.11}$$

$$\|\nu_{\Lambda^k_{i,r}} \circ g^k_{i,r}\|_{\mathrm{BV}(V^k_{i,r};\mathbb{R}^m)} \le C_{29}, \quad (i = 1, \dots, N; k = 0, 1, 2, \dots), \tag{6.12}$$

for some constant

$$C_{29} = C_{29}\big(m, \max_i (\|g_{\Lambda_i}\|_{L^1(V_{\Lambda_i};\mathbb{R}^m)} + \|\nabla g_{\Lambda_i}\|_{\mathrm{BV}(V_{\Lambda_i};\mathbb{R}^m \times z^\perp_{\Lambda_i})})\big) < \infty,$$

independent from r. (We will always explicitly indicate any dependency on r.) It follows from (6.11) and $U_{i,r} \cap U_{n,r} = \emptyset$ that

$$\big(\Lambda^k_{i,r} + B(0, \bar{s}_r - s^k_r)\big) \cap \big(\Lambda^k_{n,r} + B(0, \bar{s}_r - s^k_r)\big) = \emptyset, \quad (i \ne n; k = 0, 1, 2, \dots), \tag{6.13}$$

Moreover, we may again split $U_{i,r} \setminus \widehat{Z}_r$ into two halves $U^{k,\pm}_{i,r}$ by $\Lambda^k_{i,r}$, $(k = 0, 1, 2, \dots)$, signs chosen consistently with $U^\pm_{i,r}$.

We next want to extend u from both sides of $\Lambda_{i,r}$ to all of $U_{i,r}$. Indeed, Proposition 4.2 provides extensions $v^{(\pm)}_{i,r} \in W^{1,\infty}(U_{i,r}; \mathbb{R}^K)$ of $u|U^\pm_{i,r} \in W^{1,\infty}(U^\pm_{i,r}; \mathbb{R}^K)$, satisfying

$$\|v^{(\pm)}_{i,r}\|_{L^\infty(U_{i,r};\mathbb{R}^K)} \le \|u\|_{L^\infty(U^\pm_{i,r};\mathbb{R}^K)} \quad \text{and} \quad \|v^{(\pm)}_{i,r}\|_{W^{1,\infty}(U_{i,r};\mathbb{R}^K)} \le C_{17}\|u\|_{W^{1,\infty}(U^\pm_{i,r};\mathbb{R}^K)} \tag{6.14}$$

for some $C_{17} = C_{17}(L, m, u)$. Moreover

$$\mathcal{L}^m(A_{i,r}) = 0 \text{ for } A_{i,r} = \{x \in U_{i,r} \mid v^{(+)}_{i,r}(x) = v^{(-)}_{i,r}(x)\}. \tag{6.15}$$

Since $V_{\Lambda^k_{i,r}}$ is polyhedral and hence has Lipschitz boundary, by (6.14) and Proposition 4.1 (after a trivial rotation of the domain), $v^{(\pm)}_{i,r}$ has a trace on $\Lambda^k_{i,r}$, satisfying

$$\|v^{(\pm)}_{i,r} \circ g^k_{i,r}\|_{W^{1,\infty}(V_{\Lambda^k_{i,r}};\mathbb{R}^K)} \le C_{15}\|v^{(\pm)}_{i,r}\|_{W^{1,\infty}(U_{i,r};\mathbb{R}^K)} \le C_{30} \tag{6.16}$$

for some constants $C_{15} = C_{15}(L', m-1)$ and $C_{30} = C_{30}(u, m, \{\Lambda_i\}_{i=1}^N)$. From the construction of $U_{i,r}$ it can be easily observed that $\mathcal{H}^{m-1}(\Lambda_i \cap \partial U_{i,r}) = 0$. Because $v^{(\pm)}_{i,r} \in W^{1,\infty}(U_{i,r}) \subset C(U_{i,r})$, referring to Proposition 2.1 it hence follows from (6.9) that

$$v^{(\pm)}_{i,r} \mathcal{H}^{m-1} \llcorner \Lambda^k_{i,r} \stackrel{*}{\rightharpoonup} v^{(\pm)}_{i,r} \mathcal{H}^{m-1} \llcorner \Lambda_i \setminus \widehat{Z}_r \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m; \mathbb{R}^K). \tag{6.17}$$

The next step is to choose some $y_r \in Q_0$ with desirable properties. Let us set $\widetilde{J}_r^k := \bigcup_{i=1}^N \Lambda_{i,r}^k$ and begin by observing that Lemma 5.3 provides a constant $C_{31} = C_{31}(J, N, m, \Omega)$ such that

$$\int_{Q_0} \sum_{h_\ell \le r} \mathcal{H}^{m-1}\big((J' \cup \partial\Omega) \cap (Z_r^y + B(0, 2h_\ell)) \setminus Z_r^y\big)\, dy \le C_{31} \sum_{h_\ell \le r} h_\ell, \quad (J' = J, \widetilde{J}_r^0, \widetilde{J}_r^1, \widetilde{J}_r^2, \ldots).$$

Likewise from Lemma 5.4 it follows that

$$\int_{Q_0} \mathcal{H}^{m-2}((J' \cup \partial\Omega) \cap \partial Z_r^y)\, dy \le C_{24}, \quad (J' = J, \widetilde{J}_r^0, \widetilde{J}_r^1, \widetilde{J}_r^2, \ldots).$$

for some constant $C_{24} = C_{24}(J, N, m, \Omega)$. Application of Fatou's inequality with $J' = \widetilde{J}_r^k$, $(k = 0, 1, 2, \ldots)$, now gives

$$I_1 := \int_{Q_0} \liminf_{k\to\infty} \left(M_u \mathcal{H}^{m-2}((\widetilde{J}_r^k \cup \partial\Omega) \cap \partial Z_r^y) + \frac{\sum_{h_\ell \le r} \mathcal{H}^{m-1}\Big((\widetilde{J}_r^k \cup \partial\Omega) \cap (Z_r^y + B(0, 2h_\ell)) \setminus Z_r^y\Big)}{\sum_{h_\ell \le r} h_\ell} \right) dy \le C_{32}$$

for $C_{32} = C_{31} + M_u C_{24}$. Likewise setting $J' = J$ gives

$$I_2 := \int_{Q_0} \left(M_u \mathcal{H}^{m-2}((J \cup \partial\Omega) \cap \partial Z_r^y) + \frac{\sum_{h_\ell \le r} \mathcal{H}^{m-1}\Big((J \cup \partial\Omega) \cap (Z_r^y + B(0, 2h_\ell)) \setminus Z_r^y\Big)}{\sum_{h_\ell \le r} h_\ell} \right) dy \le C_{32}.$$

It follows that

$$I_1 + I_2 \le C_{33}$$

for some constant $C_{33} = C_{33}(u, N)$ independent of $r \in (0, 1)$. Consequently there is a subset $\widehat{Q}_r \subset Q_0$ with measure $\mathcal{L}^m(\widehat{Q}_r) > 0$, such that choosing any $y_r \in \widehat{Q}_r$, and denoting $F_r := F_r^{y_r}$ and $Z_r := Z_r^{y_r} = F_r + rQ$, we have

$$M_u \mathcal{H}^{m-2}((J \cup \partial\Omega) \cap \partial Z_r^y) + \frac{\sum_{h_\ell \le r} \mathcal{H}^{m-1}\Big((J \cup \partial\Omega) \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r\Big)}{\sum_{h_\ell \le r} h_\ell} \le C_{33}, \quad \text{and} \tag{6.18}$$

$$\liminf_{k\to\infty} \left(M_u \mathcal{H}^{m-2}((\widetilde{J}_r^k \cup \partial\Omega) \cap \partial Z_r^y) + \frac{\sum_{h_\ell \le r} \mathcal{H}^{m-1}\Big((\widetilde{J}_r^k \cup \partial\Omega) \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r\Big)}{\sum_{h_\ell \le r} h_\ell} \right) \le C_{33}. \tag{6.19}$$

Let now $\alpha_r \in [-M_u, M_u]^K$ be such that

$$\mathcal{H}^{m-1}(\{x \in \partial Z_r \mid w(x) = \alpha_r\}) = 0 \quad \text{for all} \quad w = u, v_{i,r}^{(+)}, v_{i,r}^{(-)},\ i = 1, \ldots, N.$$

(The existence of α_r is a consequence of the formula $\int_\Omega f\, d\mu = \int_0^M \mu(\{f > t\})\, dt = \int_0^M \mu(\{f \ge t\})\, dt$ for bounded Borel $f : \Omega \to [0, M]$. Here $\Omega = \partial Z_r$, $\mu = \mathcal{H}^{m-1}$.)

We are then finally in the position to define the approximations

$$w_r^k(x) := \begin{cases} \alpha_r, & x \in Z_r \cap \Omega, \\ v_{i,r}^{(\pm)}(x), & x \in U_{i,r}^{k,\pm} \setminus Z_r, \\ u(x), & \text{otherwise in } \Omega. \end{cases}$$

We want to show that $w_r^k \in \mathcal{A}(\Omega)$, and that $\{w_r^k\}_{k=0}^\infty$ converge in a suitable sense to

$$v_r(x) := \begin{cases} \alpha_r, & x \in Z_r \cap \Omega, \\ u(x), & \text{otherwise in } \Omega. \end{cases}$$

Then showing that v_r converges to u as $r \searrow 0$, a diagonal sequence $\{u^i = w_{r_i}^{k_i}\}_{i=0}^\infty$, $(r_i \searrow 0,\ k_i \to \infty)$, will satisfy the claim of the lemma.

Regarding the claim that $w_r^k \in \mathcal{A}(\Omega; \mathbb{R}^K)$, clearly $w_r^k \in W^{1,\infty}(\Omega \setminus J_r^k; \mathbb{R}^K)$ for the polyhedral set

$$J_r^k := (\widetilde{J}_r^k \setminus Z_r) \cup (\partial Z_r \cap \Omega).$$

Observe also that $J_{w_r^k} \setminus Z_r = J_r^k \cap A_{i,r} \setminus Z_r$, so that, thanks to (6.15), we have $\mathcal{H}^{m-1}((J_r^k \setminus J_{w_r^k}) \setminus Z_r) = 0$. Due to the choice of α_r, also $\mathcal{H}^{m-1}((J_r^k \setminus J_{w_r^k}) \cap Z_r) = 0$. Together these yield

$$\mathcal{H}^{m-1}(J_r^k \setminus J_{w_r^k}) = 0. \tag{6.20}$$

This takes care of condition (i) of Definition (5.1). Condition (iv) will be shown during the course of the convergence proof in Step 3. The remaining conditions follow from the construction above; to force condition (iii), we have to break each face of ∂Z_r into multiple graphs by $\{\Gamma_{i,r}^k\}_{i=1}^N$. Since the graphs $\Gamma_{i,r}^k$ are piecewise affine, condition (v) is retained.

Step 2: Convergence of v_r to u We have to show the convergences (6.1)–(6.6) for $u^i = v_{r_i}$, ($r_i \searrow 0$). First of all, we observe that v_r has its jump set J_{v_r} concentrated on

$$J_r := (J \setminus Z_r) \cup (\partial Z_r \cap \Omega).$$

By construction we have $J_{v_r} \setminus Z_r = J_u \setminus Z_r$ and $J_r \setminus Z_r = J \setminus Z_r$. Thus by Definition 5.1(i), $\mathcal{H}^{m-1}((J_r \setminus J_{v_r}) \setminus Z_r) = 0$. Due to the choice of α_r we thus further obtain

$$\mathcal{H}^{m-1}(J_r \setminus J_{v_r}) = 0. \tag{6.21}$$

Next we recall from Lemma 5.2 that there are at most $K_r \leq C_{19} r^{2-m}$ points of $ry_r + r\mathbb{Z}^m$ in F_r for some constant $C_{19} = C_{19}(J)$. Thus we deduce

$$\mathcal{L}^m(Z_r) \leq K_r \mathcal{L}^m(rQ) \leq C_{19} r^2. \tag{6.22}$$

Since $v_r = u$ on $\Omega \setminus Z_r$, this clears the convergences $v_r \to u$ strongly in $L^2(\Omega; \mathbb{R}^K)$, and $\nabla v_r \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^{K \times m})$ as $r \searrow 0$. The convergence

$$\mathcal{H}^{m-1}(J_{v_r}) \to \mathcal{H}^{m-1}(J_u)$$

follows from the following two observations. Firstly $\mathcal{H}^{m-1}(J_u \setminus J_{v_r}) = \mathcal{H}^{m-1}(J_u \cap \operatorname{int} Z_r)$ by construction. But $\mathcal{H}^{m-1}(J_u \cap \operatorname{int} Z_r) \to 0$ as $r \searrow 0$ by (6.22) and the (obvious) upper Ahlfors regularity of J_u. Secondly, $\mathcal{H}^{m-1}(J_{v_r} \setminus J_u) \leq \mathcal{H}^{m-1}(\partial Z_r) \to 0$ due to the estimate

$$\mathcal{H}^{m-1}(\partial Z_r) \leq K_r \mathcal{H}^{m-1}(\partial(rQ)) \leq C_{19} r^{2-m} \cdot 2m r^{m-1} = C_{34} r. \tag{6.23}$$

Since $v_r = u$ on $\Omega \setminus Z_r$, and $u \in L^\infty_{M_u}(\Omega; \mathbb{R}^K)$, we have $|T_\psi v_r - T_\psi u| \leq c_\psi \mathcal{H}^{m-1} \llcorner \partial Z_r$, where c_ψ is the maximum of ψ on the compact set $\operatorname{cl} \Omega \times \operatorname{cl} B(0, M_u) \times \operatorname{cl} B(0, M_u) \times S^{m-1}$. Minding (6.23), it follows that $T_\psi v_r \overset{*}{\rightharpoonup} T_\psi u$ weakly* in $\mathcal{M}(\mathbb{R}^m)$, ($\psi \in \mathcal{F}$), and, similarly, $D^j v_r \overset{*}{\rightharpoonup} D^j u$ weakly* in $\mathcal{M}(\mathbb{R}^m; \mathbb{R}^{K \times m})$.

We still have to show $\eta(T_\psi v_r) \to \eta(T_\psi u)$ for any $\psi \in \mathcal{F}$. We begin by studying $\eta_\ell(T_\psi v_r)$ for indices ℓ with $h_\ell > r$. Firstly, we observe that

$$|T_\psi v_r| \llcorner (J \setminus Z_r) = |T_\psi u| \llcorner (J \setminus Z_r) \quad \text{and} \quad |T_\psi v_r| \llcorner Z_r \leq c_\psi \mathcal{H}^{m-1} \llcorner \partial Z_r.$$

Thus an application of (6.23) and Lemma 3.1(i) yields the estimate

$$\eta_\ell(T_\psi v_r) \leq \eta_\ell(T_\psi v_r \llcorner J \setminus Z_r) + 2|T_\psi v_r \llcorner Z_r|(\Omega) \leq \eta_\ell(T_\psi u) + 2c_\psi C_{34} r,$$

and summing over $h_\ell > r$ gives

$$\sum_{h_\ell > r} \eta_\ell(T_\psi v_r) \le \sum_{h_\ell > r} \eta_\ell(T_\psi u) + 2c_\psi C_{34} \sum_{h_\ell > r} r. \tag{6.24}$$

We then study $\eta_\ell(T_\psi v_r)$ for indices ℓ with $h_\ell \le r$. Letting $D(x;\mu) := |\mu|(\tau_x f_\ell) - |\mu(\tau_x f_\ell)|$, we may write

$$\eta_\ell(T_\psi v_r) = \int_{\mathbb{R}^m} D(x; T_\psi v_r)\,dx = \int_A D(x; T_\psi v_r)\,dx + \int_B D(x; T_\psi v_r)\,dx, \tag{6.25}$$

for $A := Z_r + B(0, h_\ell)$ and $B := \mathbb{R}^m \setminus A$. The second integral we may approximate

$$\int_B D(x; T_\psi v_r)\,dx = \int_B D(x; T_\psi u)\,dx \le \int_{\mathbb{R}^m} D(x; T_\psi u)\,dx = \eta_\ell(T_\psi u). \tag{6.26}$$

We then consider the integral over $A = Z_r + B(0, h_\ell)$. First of all, since $\operatorname{supp} f_\ell \subset B(0, h_\ell)$, we deduce that

$$\int_A D(x; T_\psi v_r)\,dx \le \eta_\ell(T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell))). \tag{6.27}$$

We intend to use Proposition 3.3, towards which end we need to estimate $\operatorname{Sp}(T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell))$. Observing that

$$|T_\psi v_r| \llcorner (Z_r + B(0, 2h_\ell)) \le c_\psi(\mathcal{H}^{m-1} \llcorner \partial Z_r + \mathcal{H}^{m-1} \llcorner J \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r), \tag{6.28}$$

it suffices to study

$$\mu_{r,\ell} := \mathcal{H}^{m-1} \llcorner \partial Z_r + \mathcal{H}^{m-1} \llcorner J \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r.$$

By Lemma 5.6 we indeed have the bound

$$\operatorname{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_{r,\ell}) \le \mathcal{H}^{m-1}\big(J \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r\big) + C_{28} h_\ell \tag{6.29}$$

for $C_{28} = C_{28}(J)$ and the collection

$$\mathcal{G}_{r,\ell} := \{\Gamma^x_\ell := \partial Z_r \cap B(x, h_\ell) \mid B(x, h_\ell) \text{ intersects at most one face of } Z_r\} \tag{6.30}$$

of Lipschitz graphs satisfying (3.13). An application of (6.18) yields

$$\sum_{h_\ell \le r} \operatorname{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_{r,\ell}) \le C_{35} \sum_{h_\ell \le r} h_\ell \tag{6.31}$$

for some $C_{35} = C_{35}(u, J, N)$.

Writing

$$\theta_{\psi,r,\ell}\mu_{r,\ell} := \psi(\cdot, v_r^+, v_r^-, \nu_{J_{v_r}})\mathcal{H}^{m-1} \llcorner \big(J_{v_r} \cap (Z_r + B(0, 2h_\ell))\big) = T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell)),$$

we now have by Proposition 3.3 for some constant $C_{36} = C_{36}(L, m, \alpha)$ that

$$\begin{aligned} \eta_\ell(T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell)) &\le C_{36} h_\ell \|\theta_{\psi,r,\ell}\|_{\mathrm{BV}(\mathcal{G}_{r,\ell})} + \operatorname{Sp}_\ell(\theta_{\psi,r,\ell}\mu_{r,\ell}; \mathcal{G}_{r,\ell}) \\ &\le C_{36} h_\ell \left(\sup_{\{\Gamma\}} \sum_\Gamma \|\theta_{\psi,r,\ell} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \right) + c_\psi \operatorname{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_{r,\ell}). \end{aligned} \tag{6.32}$$

The supremum is taken over finite disjoint subcollections of $\mathcal{G}_{r,\ell}$. Recalling (6.30), this amounts to simply taking the sum over all the faces (see Definition 5.3) of Z_r. Let us denote this collection by $\mathcal{V}_r$. Extending u and v by zero outside Ω, for them to be fully defined on all $\Gamma \in \mathcal{V}_r$, we then have to bound

$$\sum_{\Gamma \in \mathcal{V}_r} \|\theta_{\psi,r,\ell} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} = \sum_{\Gamma \in \mathcal{V}_r} \|\psi(\cdot, v_r^+ \circ g_\Gamma, v_r^- \circ g_\Gamma, \nu_\Gamma \circ g_\Gamma)\|_{\mathrm{BV}(V_\Gamma)}.$$

Since ψ is C^1, it is Lipschitz on the compact set $\operatorname{cl}\Omega \times \operatorname{cl} B(0, M_u) \times \operatorname{cl} B(0, M_u) \times S^{m-1}$, and we may apply the BV chain rule [3]. We thus only have to bound $\|\nu_\Gamma \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}$ and $\|v_r^\pm \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}$ for $\Gamma \in \mathcal{V}_r$. Since each $\Gamma \in \mathcal{V}_r$ is a face of ∂Z_r, we find that ν_Γ is constant with

$$\sum_{\Gamma \in \mathcal{V}_r} \|\nu_\Gamma \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} = \sum_{\Gamma \in \mathcal{V}_r} \mathcal{H}^{m-1}(g_\Gamma(V_\Gamma)) = \mathcal{H}^{m-1}(\partial Z_r).$$

This is indeed bounded due to (6.23). On the other hand, the definition $v_r = (1-\chi_{Z_r})u + \alpha_r \chi_{Z_r}$ gives

$$\sum_{\Gamma \in \mathcal{V}_r} \|v_r^+ \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} + \sum_{\Gamma \in \mathcal{V}_r} \|v_r^- \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \le 2M_u \mathcal{H}^{m-1}(\partial Z_r) + \sum_{\Gamma \in \mathcal{V}_r} \|u \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}.$$

Since $u \in W^{1,\infty}(\Omega \setminus J)$ and Lipschitz continuity is preserved by traces on affine sets, we may bound

$$\|u \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \le \int_{\Gamma \cap \Omega} \|u(x)\| + \|\nabla u(x)\| \, d\mathcal{H}^{m-1}(x) + 2M_u \mathcal{H}^{m-2}((J \cup \partial\Omega) \cap \Gamma).$$

The latter term approximates the mass of the jump part of the differential. Summing over $\Gamma \in \mathcal{V}_r$ we thus obtain

$$\begin{aligned}\sum_{\Gamma \in \mathcal{V}_r} \|u \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} &\le \int_{\partial Z_r \cap \Omega} \|u(x)\| + \|\nabla u(x)\| \, d\mathcal{H}^{m-1}(x) + 2M_u \mathcal{H}^{m-2}((J \cup \partial\Omega) \cap \partial Z_r) \\ &\le \|u\|_{W^{1,\infty}(\Omega;\mathbb{R}^K)} \mathcal{H}^{m-1}(\partial Z_r \cap \Omega) + 2M_u \mathcal{H}^{m-2}((J \cup \partial\Omega) \cap \partial Z_r) \\ &\le \|u\|_{W^{1,\infty}(\Omega;\mathbb{R}^K)} C_{34} r + 2C_{33}, \quad (r \in (0,1)).\end{aligned} \tag{6.33}$$

In the final step we have applied (6.23) and (6.18). Applying this in (6.32), it now follows for some $C_{37} = C_{37}(u, N, L, m, \alpha, \Omega)$ that

$$\eta_\ell(T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell)) \le C_{37} h_\ell + c_\psi \mathrm{Sp}_\ell(\mu_{r,\ell}; \mathcal{G}_{r,\ell}). \tag{6.34}$$

Applying (6.31), we may now deduce from (6.34) for some $C_{38} = C_{38}(u, J, N, L, m, \alpha, \Omega)$ that

$$\sum_{h_\ell \le r} \eta_\ell(T_\psi v_r \llcorner (Z_r + B(0, 2h_\ell)) \le C_{38} \sum_{h_\ell \le r} h_\ell.$$

Recalling (6.25)– (6.27) it then follows that

$$\sum_{h_\ell \le r} \eta_\ell(T_\psi v_r) \le \sum_{h_\ell \le r} \eta_\ell(T_\psi u) + C_{38} \sum_{h_\ell \le r} h_\ell, \quad (h_\ell \le r). \tag{6.35}$$

The estimate (6.24) for the cases $h_\ell > r$ together with (6.35) now yields

$$\eta(T_\psi v_r) \le \eta(T_\psi u) + C_{39} \sum_{\ell=0}^{\infty} \min\{h_\ell, r\}, \quad (\psi \in \mathcal{F}),$$

for some $C_{39} = C_{39}(u, J, N, L, m, \alpha, \Omega, \mathcal{F})$. Recalling the condition (3.1) in the Definition 3.1 of a regular nested sequence of functions, the sum tends to zero as $r \searrow 0$. Since $T_\psi v_r \overset{*}{\rightharpoonup} T_\psi u$ and η is known from Theorem 3.1 to be lower-semicontinuous with respect to weak* convergence, this gives $\eta(T_\psi v_r) \to \eta(T_\psi u)$. The proof of properties and convergence of the preliminary approximations $\{v_r\}_{r \in (0,1)}$ can thus be concluded.

Step 3: Convergence of w_r^k to v_r We now need to show that $\{w_r^k\}_{k=0}^\infty$ approximate v_r sufficiently close to the senses (6.1)–(6.6), in that a converging diagonal sequence can be constructed.

We begin by observing that (6.11) and the construction of the functions w_r^k and v_r yield

$$\|w_r^k - v_r\|^2_{L^2(\Omega;\mathbb{R}^K)} = \int_\Omega \chi_{J+B(0,s_r^k)} \|w_r^k(x) - v_r(x)\|^2 \, dx,$$

where $\mathcal{L}^m(J + B(0, s_r^k)) \to 0$ as $k \to \infty$. Minding that

$$\|w_r^k\|_{L^2(\Omega;\mathbb{R}^K)} \le \|u\|_{L^2(\Omega;\mathbb{R}^K)} + \sum_{i=1}^N \big(\|v_{i,r}^{(+)}\|_{L^2(U_{i,r};\mathbb{R}^K)} + \|v_{i,r}^{(-)}\|_{L^2(U_{i,r};\mathbb{R}^K)}\big)$$

is bounded, it therefore follows that $w_r^k \to v_r$ strongly in $L^2(\Omega;\mathbb{R}^K)$. Analogously we get $\nabla w_r^k \to \nabla v_r$ strongly in $L^2(\Omega;\mathbb{R}^{K\times m})$.

Let us then fix $\psi \in \mathcal{F}$. We now have to study in what sense $\eta(T_\psi w_r^k)$ approximates $\eta(T_\psi v_r)$ as $k \to \infty$. We begin by studying $\eta_\ell(T_\psi w_r^k)$ for indices ℓ with $h_\ell \le \bar{s}_r/3$ with the intent of applying Proposition 3.3 again. Then, observing that $|T_\psi w_r^k| \le c_\psi \lambda_r^k$ for

$$\lambda_r^k := \mathcal{H}^{m-1} \llcorner J_r^k = \mathcal{H}^{m-1} \llcorner \partial Z_r + \mathcal{H}^{m-1} \llcorner (\widetilde{J}_r^k \setminus Z_r),$$

it suffices to calculate $\mathrm{Sp}_\ell(\lambda_r^k; \mathcal{G}_{r,\ell}^k)$ for some collections $\mathcal{G}_{r,\ell}^k$ of Lipschitz graphs $\Gamma_\ell^x = \Gamma_{r,\ell}^{k,x}$ yet to be determined. We may further assume that k is large enough that

$$(\bar{s}_r - s_r^k) \ge (2/3)\bar{s}_r \ge 2h_\ell.$$

As in Step 2, we split the integral in (3.5) as

$$\mathrm{Sp}_\ell(\lambda_r^k; \mathcal{G}_{r,\ell}^k) = \int_A |\lambda_r^k \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx + \int_B |\lambda_r^k \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx, \tag{6.36}$$

for $A := Z_r + B(0, h_\ell)$ and $B := \mathbb{R}^m \setminus A$. If $x \in B$, then from (6.13) and $(\bar{s}_r - s_r^k) \ge 2h_\ell$, we observe that the ball $B(x, h_\ell)$ intersects at most one of the graphs $\Lambda_{1,r}^k, \ldots, \Lambda_{N,r}^k$. If $B(x, h_\ell)$ intersects, say, $\Lambda_{i,r}^k$, we then take

$$\Gamma_\ell^x = \big(B(x, h_\ell) + \mathbb{R} z_{\Lambda_{i,r}^k}\big) \cap \Lambda_{i,r}^k.$$

Otherwise, if $J_r^k \cap B(x, h_\ell) = \emptyset$, we take $\Gamma_\ell^x = \emptyset$. In either case, we have $J_r^k \cap O_\ell^x \setminus \Gamma_\ell^x = \emptyset$, so

$$\int_B |\lambda_r^k \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx = 0. \tag{6.37}$$

We define the collections $\widetilde{\mathcal{G}}_{r,\ell}^k := \{\Gamma_\ell^x \mid x \in B\}$, $(2h_\ell \le \bar{(}2/3)s_r \le (\bar{s}_r - s_r^k))$. Each $\Gamma \in \widetilde{\mathcal{G}}_{r,\ell}^k$ is a Lipschitz graph of constant at most $L'(r)$ and satisfies (3.13).

With regard to $A = Z_r + B(0, h_\ell)$, an application of Lemma 3.1(ii) gives

$$\begin{aligned}\int_A |\lambda_r^k \llcorner O_\ell^x \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx &\le \int |\lambda_r^k \llcorner (Z_r + B(0, 2h_\ell)) \setminus \Gamma_\ell^x|(\tau_x f_\ell)\, dx \\ &= \mathrm{Sp}_\ell(\lambda_r^k \llcorner (Z_r + B(0, 2h_\ell)); \mathcal{G}_{r,\ell}^k).\end{aligned} \tag{6.38}$$

Lemma 5.6 this time gives

$$\mathrm{Sp}_\ell(\lambda_r^k \llcorner (Z_r + B(0, 2h_\ell)); \mathcal{G}_{r,\ell}) \le \mathcal{H}^{m-1}\big(\widetilde{J}_r^k \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r\big) + C_{28} h_\ell$$

for exactly the same collections $\mathcal{G}_{r,\ell}$, $(\ell = 0, 1, 2, \ldots)$, as in Step 2. Setting $\mathcal{G}^k_{r,\ell} := \mathcal{G}_{r,\ell} \cup \widetilde{\mathcal{G}}^k_{r,\ell}$ and recalling (6.36)–(6.38), it thus follows that

$$\mathrm{Sp}_\ell(\lambda^k_r; \mathcal{G}^k_{r,\ell}) \leq \mathcal{H}^{m-1}\big(\widetilde{J}^k_r \cap (Z_r + B(0, 2h_\ell)) \setminus Z_r\big) + C_{28} h_\ell.$$

By application of (6.19), we therefore obtain for some $C_{35} = C_{35}(u, J, N)$ that

$$\liminf_{k\to\infty} \sum_{h_\ell \leq \bar{s}_r/3} \mathrm{Sp}_\ell(\lambda^k_r; \mathcal{G}^k_{r,\ell}) \leq C_{35} \sum_{h_\ell \leq r} h_\ell. \tag{6.39}$$

It is now possible to apply Proposition 3.3 on

$$T_\psi w^k_r = \vartheta^k_{\psi,r} \lambda^k_r := \psi(\cdot, (w^k_r)^+, (w^k_r)^-, \nu_{J_{w^k_r}}) \chi_{J_{w^k_r}} \lambda^k_r.$$

This yields for some $C_{40} = C_{40}(L', m, \alpha)$ the estimate

$$\begin{aligned} \eta_\ell(T_\psi w^k_r) &\leq C_{40} h_\ell \|\vartheta^k_{\psi,r}\|_{\mathrm{BV}(\mathcal{G}^k_{r,\ell})} + \mathrm{Sp}_\ell(\vartheta^k_{\psi,r} \lambda^k_r; \mathcal{G}^k_{r,\ell}) \\ &\leq C_{40} h_\ell \left(\sup_{\{\Gamma\}} \sum_\Gamma \|\vartheta^k_{\psi,r} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \right) + c_\psi \mathrm{Sp}_\ell(\lambda^k_r; \mathcal{G}^k_{r,\ell}). \end{aligned} \tag{6.40}$$

The supremum is taken over finite disjoint subcollections of $\mathcal{G}^k_{r,\ell}$. Minding the construction of $\mathcal{G}^k_{r,\ell}$, this amounts to simply taking all the faces $\Gamma \in \mathcal{V}_r$ of Z_r along with $\Lambda^k_{i,r}$ for $i = 1, \ldots, N$. With r fixed, we thus have to bound $\sum_{\Gamma \in \mathcal{V}_r \cup \{\Lambda^k_{i,r}\}_{i=1}^N} \|\vartheta^k_{\psi,r} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}$. With the additional help of (6.14) and (6.19) for estimates within $U^{k,\pm}_{i,r}$ (where $w^k_r = v_{i,r}$), we can similarly to (6.33) in Step 2, bound

$$\sum_{\Gamma \in \mathcal{V}_r} \|\vartheta^k_{\psi,r} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \leq C_{41} = C_{41}(u, J, N)$$

As for the remaining sum over the surfaces $\Lambda^k_{i,r}$, $(i = 1, \ldots, N)$, we have

$$\sum_{\Gamma = \Lambda^k_{1,r}, \ldots, \Lambda^k_{N,r}} \|\vartheta^k_{\psi,r} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} = \sum_{\Gamma = \Lambda^k_{1,r}, \ldots, \Lambda^k_{N,r}} \|\psi(\cdot, (w^k_r)^+ \circ g_\Gamma, (w^k_r)^- \circ g_\Gamma, \nu_{J_{w^k_r}} \circ g_\Gamma)\|_{\mathrm{BV}(V_\Gamma)},$$

since ψ is C^1 on the compact set $\mathrm{cl}\,\Omega \times \mathrm{cl}\,B(0, M_u) \times \mathrm{cl}\,B(0, M_u) \times S^{m-1}$, we may again apply the BV chain rule and only have to bound $\|\nu_{J_{w^k_r}} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}$ and $\|(w^k_r)^\pm \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)}$ for $\Gamma = \Lambda^k_{i,r}$, $(i = 1, \ldots, N;$ $k = 0, 1, 2, \ldots)$. Such bounds are given by the estimates (6.12) and (6.16). Thus

$$\sum_\Gamma \|\vartheta^k_{\psi,r} \circ g_\Gamma\|_{\mathrm{BV}(V_\Gamma)} \leq C_{42} = C_{42}(u, m, J).$$

We now obtain from (6.40) for some $C_{43} = C_{43}(L', m, \alpha, \Omega, \psi, J)$ the estimate

$$\eta_\ell(T_\psi w^k_r) \leq C_{43} h_\ell + c_\psi \mathrm{Sp}_\ell(\lambda^k_r; \mathcal{G}^k_{r,\ell}).$$

Summing over $h_\ell \leq \bar{s}_r/3$ and recalling (6.39) and the finiteness of $\mathcal{F}$ yields

$$\liminf_{k\to\infty} \sum_{\psi \in \mathcal{F}} \left(\sum_{h_\ell \leq \bar{s}_r/3} \eta_\ell(T_\psi w^k_r) \right) \leq C_{44} \sum_{h_\ell \leq r} h_\ell \tag{6.41}$$

for some $C_{44} = C_{44}(u, J, N, L', m, \alpha, \Omega, \mathcal{F})$. For $h_\ell > \bar{s}_r/3$, we have the rough bound

$$\eta_\ell(T_\psi w^k_r) \leq |T_\psi w^k_r|(\Omega) \leq c_\psi \mathcal{H}^{m-1}(J_{w^k_r}), \quad (\psi \in \mathcal{F}).$$

It follows that

$$\liminf_{k\to\infty} \sum_{\psi\in\mathcal{F}} \eta(T_\psi w_r^k) \le C_{45}(r) = C_{45}(u, J, N, L', m, \alpha, r, \Omega, \mathcal{F}),$$

so, after passing to an unrelabelled subsequence, we have for any fixed $r \in (0,1)$ that

$$\sup_k \eta(T_\psi w_r^k) < \infty, \quad (\psi \in \mathcal{F}). \tag{6.42}$$

Next we intend to apply Lemma 4.3 to show the weak* convergence of $\{T_\psi w_r^k\}_{k=0}^\infty$ to $T_\psi v_r$. We begin by deducing from (6.18) that $\mathcal{H}^{m-1}(J \cap \partial Z_r) = 0$. Thus Proposition 2.1 and (6.9) give

$$\mathcal{H}^{m-1} \llcorner J_r^k \setminus Z_r \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner J_r \setminus Z_r \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m).$$

As $\partial Z_r \cap J_r = \partial Z_r \cap J_r^k = \partial Z_r$, $(k = 0, 1, 2, \ldots)$, it follows that

$$\mathcal{H}^{m-1} \llcorner J_r^k \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner J_r \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m).$$

Recalling (6.20), (6.21), we thus have

$$\mathcal{H}^{m-1} \llcorner J_{w_r^k} \overset{*}{\rightharpoonup} \mathcal{H}^{m-1} \llcorner J_{v_r} \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m).$$

By the convergence of $\{w_r^k\}_{k=0}^\infty$ to v_r in $H^2(\Omega)$, shown in the beginning of the present step, the trace of w_r^k on ∂Z_r converges to that of v_r in L^1. Therefore (6.10) and (6.17) yield analogously to the above that

$$\nu_{J_{w_r^k}} \mathcal{H}^{m-1} \llcorner J_{w_r^k} \overset{*}{\rightharpoonup} \nu_{J_{v_r}} \mathcal{H}^{m-1} \llcorner J_{v_r} \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m; S^{m-1}), \quad \text{and} \tag{6.43}$$

$$(w_r^k)^\pm \mathcal{H}^{m-1} \llcorner J_{w_r^k} \overset{*}{\rightharpoonup} v_r^\pm \mathcal{H}^{m-1} \llcorner J_{v_r} \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^m; \mathbb{R}^K). \tag{6.44}$$

We may assume that $\mathcal{F}$ includes the functions

$$\begin{aligned}
\psi_i^\nu &: (x, u^+, u^-, \nu) \mapsto \nu_i \quad \text{(for Lemma 4.3)},\\
\psi_i^\pm &: (x, u^+, u^-, \nu) \mapsto (u^\pm)_i \quad \text{(for Lemma 4.3)},\\
\psi_{i,n} &: (x, u^+, u^-, \nu) \mapsto [(u^+ - u^-)_i \nu_n], \quad \text{and}\\
\psi_{\mathcal{H}} &: (x, u^+, u^-, \nu) \mapsto \|\nu\| \equiv 1, \quad (i, n = 1, \ldots, m).
\end{aligned}$$

It now follows from (6.42)–(6.44), and Lemma 4.3, after possibly passing to a subsequence, unrelabelled, that both $T_\psi w_r^k \overset{*}{\rightharpoonup} T_\psi v_r$ and $|T_\psi w_r^k| \overset{*}{\rightharpoonup} |T_\psi v_r|$ in $\mathcal{M}(\mathbb{R}^m)$ for all $\psi \in \mathcal{F}$. By the inclusion of $\psi_{i,n}$ in $\mathcal{F}$, $(i, n = 1, \ldots, m)$, it follows that $D^j w_r^k \overset{*}{\rightharpoonup} D^j v_r$ as well as $|D^j w_r^k|(\Omega) \to |D^j v_r|(\Omega)$. Moreover, by the inclusion of $\psi_{\mathcal{H}}$ in $\mathcal{F}$, we get $\mathcal{H}^{m-1}(J_{w_r^k}) \to \mathcal{H}^{m-1}(J_v)$.

We must still study the convergence of $\eta(T_\psi w_r^k)$ to $\eta(T_\psi v_r)$. As we have shown above that $T_\psi w_r^k \overset{*}{\rightharpoonup} T_\psi v_r$, and $|T_\psi w_r^k| \overset{*}{\rightharpoonup} |T_\psi v_r|$ in $\mathcal{M}(\mathbb{R}^m)$ it follows from Theorem 3.1 that $\eta_\ell(T_\psi w_r^k) \to \eta_\ell(T_\psi v_r)$, $(\ell = 0, 1, 2, \ldots)$. By the lower-semicontinuity of η and, respectively, (6.41), it follows that by choosing $k(r)$ large enough, we can ascertain the lower and upper bounds

$$\eta(T_\psi v_r) - 2C_{44} \sum_{h_\ell \le r} h_\ell \le \eta(T_\psi w_r^{k(r)}) \le \eta(T_\psi v_r) + 2C_{44} \sum_{h_\ell \le r} h_\ell, \quad (\psi \in \mathcal{F}). \tag{6.45}$$

The sum $\sum_{h_\ell \le r} h_\ell$ tends to zero as $r \searrow 0$, so $\eta(T_\psi w_r^{k(r)}) - \eta(T_\psi v_r) \to 0$ as $r \searrow 0$.

Summarising, taking $k(r)$ sufficiently large, we can thus ask that (6.45) holds as do

$$\mathcal{H}^{m-1}(J_{v_r}) - r \le \mathcal{H}^{m-1}(J_{w_r^{k(r)}}) \le \mathcal{H}^{m-1}(J_{v_r}) + r$$

along with

$$\|v_r - w_r^{k(r)}\|_{L^2(\Omega;\mathbb{R}^K)} \le r, \quad \text{and} \quad \|\nabla v_r - \nabla w_r^{k(r)}\|_{L^2(\Omega;\mathbb{R}^{K\times m})} \le r.$$

Metricising the weak topology on $\mathcal{M}(\mathbb{R}^m)$ with d^*, we can also ensure that

$$d^*(D^j v_r, D^j w_r^{k(r)}) \le r, \quad \text{and} \quad d^*(T_\psi v_r, T_\psi w_r^{k(r)}) \le r, \quad (\psi \in \mathcal{F}).$$

Minding the preliminary approximation results of Step 2, we thus obtain the desired convergences (6.1)–(6.6) for the sequence $u^i := w_{r_i}^{k(r_i)}$ given $r_i \searrow 0$. This completes the proof. □

Remark 6.1. Provided that $\mathrm{Sp}(J \cup \partial\Omega)$ is bounded, it is easy to extend the above proof to show that if $\bar{u}$ (resp. $\bar{u}^i$) is the extension of u (resp. u^i) to $\mathbb{R}^m$ by zero, then the sequence $\{\bar{u}^i\}_{i=0}^\infty$ converges to $\bar{u}$ in the senses (6.1)–(6.6) with $\Omega = \mathbb{R}^m$. (The important point is that parts of $\partial\Omega$ now are contained in J_u.) Indeed, all we have to do is to include the graphs $\Gamma_1^\Omega, \ldots, \Gamma_M^\Omega$, where $\partial\Omega = \bigcup_{i=1}^M \Gamma_i^\Omega$, among $\Lambda_1, \ldots, \Lambda_N$ in the construction of the theorem. We however do not need to cover the boundaries by jump cubes or to approximate them by polyhedral graphs as we do approximate $\Lambda_1, \ldots, \Lambda_N$. Hence there is also no need to extend u over $\Gamma_1^\Omega, \ldots, \Gamma_M^\Omega$ (as $v_{i,r}^\pm$). The only thing that we need to take worry about is the effect of the jump cubes on Sp. This is the reason why we have already included $\partial\Omega$ in the $\mathcal{H}^{m-1}$ bounds of (6.18) and (6.19); doing so was not necessary for the proof above. (Including $\partial\Omega$ in the $\mathcal{H}^{m-2}$ bounds is however necessary for bounding quantities of the form $\|\theta\|_{\mathrm{BV}(\mathcal{G}_\ell)}$ with $\Gamma_\ell^x \in \mathcal{G}_\ell$ extending outside Ω.)

7. An anisotropic variant

We next study a variant of Theorem 6.1 approximating J by jump sets with the normal field always oriented along one of the the coordinate axes. We begin with necessary additional definitions, assumptions, and lemmas.

Definition 7.1. For $\nu \in S^{m-1}$, we define the anisotropy function $\varphi(\nu) := \sum_{i=1}^m |\langle \nu, e_i\rangle| = \|\nu\|_1$. For $\mathcal{H}^{m-1}$-rectifiable J, we let $\Phi(J) := \int_J \varphi(\nu_J)\, d\mathcal{H}^{m-1}$.

The following lemma is an analogue of Lemma 4.3.

Lemma 7.1. *Let $\mathcal{F}$ be a finite collection of maps $\psi(x, u^+, u^-, \nu) = \bar{\psi}(x, u^+, u^-)\varphi(\nu)$ for some $\bar{\psi} \in C^1(\mathrm{cl}\,\Omega \times \mathbb{R}^K \times \mathbb{R}^K)$. Suppose that $\mathcal{F}$ includes the functions $\psi_\varphi : (x, u^+, u^-, \nu) \mapsto \varphi(\nu)$, and $\psi_{\varphi,i}^\pm : (x, u^+, u^-, \nu) \mapsto u_i^\pm \varphi(\nu)$, $(i = 1, \ldots, K)$. Let $\{v, w^0, w^1, w^2, \ldots\} \subset \mathrm{SBV}(\Omega; \mathbb{R}^K) \cap L_M^\infty(\Omega; \mathbb{R}^K)$ satisfy*

$$\sup_k \mathcal{H}^{m-1}(J_{w^k}) < \infty, \tag{7.1}$$

$$\sup_k \eta(T_\psi w^k) < \infty, \quad (\psi \in \mathcal{F}), \tag{7.2}$$

$$\varphi(\nu_{J_{w^k}})\mathcal{H}^{m-1} \llcorner J_{w^k} \overset{*}{\rightharpoonup} \varphi(\nu_{J_v})\mathcal{H}^{m-1} \llcorner J_v \quad \textit{weakly* in } \mathcal{M}(\Omega), \quad \textit{and} \tag{7.3}$$

$$(w^k)^\pm \varphi(\nu_{J_{w^k}})\mathcal{H}^{m-1} \llcorner J_{w^k} \overset{*}{\rightharpoonup} v^\pm \varphi(\nu_{J_v})\mathcal{H}^{m-1} \llcorner J_v \quad \textit{weakly* in } \mathcal{M}(\Omega; \mathbb{R}^K). \tag{7.4}$$

Then, after possibly moving to an unrelabelled subsequence, we have $T_\psi w^k \overset{*}{\rightharpoonup} T_\psi v$ and $|T_\psi w^k| \overset{*}{\rightharpoonup} |T_\psi v|$ for all $\psi \in \mathcal{F}$.

Proof. The claim follows similarly to Lemma 4.3; for the application of Reshetnyak's continuity theorem, we simply write for $\mu_w := (w^+, w^-, 1)\varphi(\nu) \llcorner J_w$ that

$$\begin{aligned} f(x)\psi(x, w^+, w^-, \nu)\mathcal{H}^{m-1} \llcorner J_w &= f(x)\bar{\psi}(x, w^+, w^-)\varphi(\nu)\mathcal{H}^{m-1} \llcorner J_w \\ &= f(x)\bar{\psi}(x, w^+, w^-)\frac{1}{\|(w^+, w^-, 1)\|}|\mu_w| \\ &=: \psi_f\left(x, \frac{d\mu_w}{d|\mu_w|}\right)|\mu_w|. \end{aligned}$$

□

Remark 7.1. The lemma would also go through for $\psi(x,u^+,u^-,\nu) = \sum_{i=1}^m \bar{\psi}_i(x,(u^+,u^-))\varphi_i(\nu)$ with $\varphi_i(\nu) = |\langle \nu, e_i\rangle|$, provided the weak* convergence of $((w^k)^+,(w^k)^-,1)\varphi_i(\nu_{J_{w^k}})$ to $(u^+,u^-,1)\varphi_i(\nu_{J_u})$, $(i=1,\ldots,m)$, which actually does hold in the construction below. The reason for restricting attention to $\psi(x,u^+,u^-,\nu) = \bar{\psi}_i(x,u^+,u^-)\varphi(\nu)$ is the bound (7.6) below: $\varphi_i \circ \nu_{\Lambda^k} \circ g_{\Lambda^k}$ would have to have uniformly bounded variation for a sequence of approximations $\{\Lambda^k\}_{k=0}^\infty$. This does not generally hold with Λ^k on the faces of a tightening grid.

Theorem 7.1. *Let $\Omega = \operatorname{int} Q \subset \mathbb{R}^m$. Suppose $u \in \mathcal{A}(\Omega;\mathbb{R}^K)$. Let $\mathcal{F}$ be a finite collection of maps $\psi(x,u^+,u^-,\nu) = \bar{\psi}(x,u^+,u^-)\varphi(\nu)$ for some $\bar{\psi} \in C^1(\operatorname{cl}\Omega \times \mathbb{R}^K \times \mathbb{R}^K)$. Then there exists a sequence $\{u^i\}_{i=0}^\infty \subset \mathcal{A}(\Omega;\mathbb{R}^K)$ such that each set $\widehat{J}_{u^i}$ from Definition 5.1 satisfies $\nu_{\widehat{J}_{u^i}}(x) \in \{\pm e_1,\ldots,\pm e_m\}$, (a.e. $x \in \widehat{J}_{u^i}$), and we have the convergences* (6.1)–(6.3),(6.6) *and*

$$\Phi(J_{u^i}) \to \Phi(J_u). \tag{7.5}$$

Sketch of proof. Let $\{\Lambda_i\}_{i=1}^N$ be the graphs from Definition 5.1 for u. By including in $\mathcal{F}$ the function

$$\psi_\Phi : (x,u^+,u^-,\nu) \mapsto \varphi(\nu),$$

Theorem 6.1 yields the convergence $\Phi(J_{u^i}) \to \Phi(J_u)$ for the sequence of approximations constructed therein. Consequently, minding the construction in Theorem 6.1, we may without loss of generality assume that each of the graphs Λ_i, $(i=1,\ldots,N)$ is affine.

Next we apply Theorem 6.1 a second time with a small modification. By the assumption that Λ_i, $(i = 1,\ldots,N)$, are affine, it is easy to construct approximating graphs $\Lambda_{i,r}^k$ such that $\nu_{\Lambda_{i,r}^k} \in \{e_1,\ldots,e_m\}$. As clearly $\nu_{Z_r} \in \{e_1,\ldots,e_m\}$, it follows that $\nu_{\widehat{J}_{w_r^k}} \in \{e_1,\ldots,e_m\}$.

The only problem with this kind of approximation is that we do not have the estimate (6.12), $\{\nabla g_{\Lambda_{i,r}^k}\}_{k=0}^\infty$ not generally being bounded in the BV norm. However, since $\psi \in \mathcal{F}$ only depends on ν through $\varphi(\nu)$, we do not need to bound $\|\nu_{\Lambda_{i,r}^k} \circ g_{\Lambda_{i,r}^k}\|_{\mathrm{BV}(V_{\Lambda_{i,r}^k};\mathbb{R}^m)}$, instead needing only

$$\|\varphi \circ \nu_{\Lambda_{i,r}^k} \circ g_{\Lambda_{i,r}^k}\|_{\mathrm{BV}(V_{\Lambda_{i,r}^k})} \le C_{46}. \tag{7.6}$$

But this is trivial, because $\varphi \circ \nu_{\Lambda_{i,r}^k} \equiv 1$. □

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